Solutions Single Choice 4

1. Let a, b be algebraic over $\mathbb{Q}$, such that $[\mathbb{Q}(a) : \mathbb{Q}] = 3$ and $[\mathbb{Q}(b) : \mathbb{Q}] = 5$. Then the possible degrees of $\mathbb{Q}(a, b)$ over $\mathbb{Q}$ are

(a) $[\mathbb{Q}(a, b) : \mathbb{Q}] = 3$
(b) $[\mathbb{Q}(a, b) : \mathbb{Q}] = 5$
(c) $[\mathbb{Q}(a, b) : \mathbb{Q}] = 15$
(d) All of the above.

Solution: The correct answer is (c): note that

$$[\mathbb{Q}(a, b) : \mathbb{Q}] \leqslant [\mathbb{Q}(a) : \mathbb{Q}][\mathbb{Q}(b) : \mathbb{Q}] = 15$$

On the other hand, we have that because of multiplicativity of the degree of the field extention $\mathbb{Q}(a, b) : \mathbb{Q}$ we have that $[\mathbb{Q}(a) : \mathbb{Q}] = 3$ and $[\mathbb{Q}(b) : \mathbb{Q}] = 5$ both divide $[\mathbb{Q}(a, b) : \mathbb{Q}]$.

2. Let $L : \mathbb{Q}$ be a field extention and $a, b \in L\backslash\{0\}$ such that $a + b \neq 0$. Which of the following statements is **false**?

(a) $a^2 \in \mathbb{Q}(a + b, ab) \Rightarrow a \in \mathbb{Q}(a + b, ab)$
(b) $a \in \mathbb{Q}(a + b, ab) \Rightarrow a^2 \in \mathbb{Q}(a + b, ab)$
(c) $[\mathbb{Q}(a, b) : \mathbb{Q}(a + b, ab)]$ is equal to the degree of the minimal polynomial of a over $\mathbb{Q}(a + b, ab)$.
(d) All the statements above are true.

Solution: All the statements are true. Note that $a = \frac{a^2+ab}{a+b}$, so that we get part (a). Further note that $\mathbb{Q}(a + b, ab) \subset \mathbb{Q}(a, b) = \mathbb{Q}(a + b, ab)(a) = \mathbb{Q}(a + b, ab)(b)$ and the polynomial $x^2 + (a + b)x + ab$ has roots a and b.

3. Let $M : L : K$ be field extentions and assume that a is algebraic over M, L and K. Then

(a) $m_{a,M} \mid m_{a,L}$ in $M[x]$
(b) $m_{a,L} \mid m_{a,K}$ in $L[x]$
(c) $m_{a,M} \mid m_{a,K}$ in $M[x]$
(d) All the statements above are true.

Solution: (d) is the answer: see Lecture Cor.2.10.

4. Consider a field extention $\mathbb{Q}(a, b) : \mathbb{Q}$. Which of the following statements is **false**?

(a) If $\mathbb{Q}(a, b) : \mathbb{Q}$ is algebraic, then also a and b are algebraic over $\mathbb{Q}$.
(b) If $\mathbb{Q}(a + b) : \mathbb{Q}$ and $\mathbb{Q}(ab) : \mathbb{Q}$ are algebraic, then also a and b are algebraic over $\mathbb{Q}$.

(c) If a is transcendental over $\mathbb{Q}(b)$, then a is also transcendental over $\mathbb{Q}$.

(d) If a is transcendental over $\mathbb{Q}$, then a is also transcendental over $\mathbb{Q}(b)$.

Solution: The answer is (d).

For (a) note this is true by definition of an algebraic extention. For (c), note that if a is algebraic over $\mathbb{Q}$, then a is also algebraic over $\mathbb{Q}(b)$. Taking the contrapositive of that yields part (c). For (b) note that a, b are the roots of the polynomial $x^2 + (a + b)x + ab$. Hence $[\mathbb{Q}(a) : \mathbb{Q}(a + b, ab)] \leqslant 2$. Since $[\mathbb{Q}(a + b, ab) : \mathbb{Q}] \leqslant [\mathbb{Q}(a + b) : \mathbb{Q}][\mathbb{Q}(ab) : \mathbb{Q}] < \infty$, we have that $[\mathbb{Q}(a) : \mathbb{Q}] < \infty$. Hence a is algebraic over $\mathbb{Q}$. For a counter-example for part (d) take $b := a$.

5. Let a, b be algebraic over $\mathbb{Q}$, such that the minimal polynomial of a and b both have degree 2. Then the degree of the minimal polynomial of $a + b$ is ...

(a) 2

(b) 4

(c) a divisor of 2

(d) a divisor of 4

Solution: The correct answer is (d). We have $[\mathbb{Q}(a, b) : \mathbb{Q}(b)] \leqslant [\mathbb{Q}(a) : \mathbb{Q}] = 2$, and hence $[\mathbb{Q}(a, b) : \mathbb{Q}] = [\mathbb{Q}(a, b) : \mathbb{Q}(b)][\mathbb{Q}(b) : \mathbb{Q}]$ is a divisor of $2 \cdot 2 = 4$. Since $\mathbb{Q} \subset \mathbb{Q}(a + b) \subset \mathbb{Q}(a, b)$, and the multiplicativity of the degree of a field extention, we have that $[\mathbb{Q}(a+b) : \mathbb{Q}]$ is a divisor of 4 as well.

Note that each divisor of 4 is possible:

If $a := \sqrt{2}$, $b := -\sqrt{2}$, then the minimal polynomial of $a + b$ has degree 1. If $a := \sqrt{2}$, $b := \sqrt{2}$, then the minimal polynomial of $a + b$ has degree 2. If $a := \sqrt{2}$, $b := \sqrt{3}$, then the minimal polynomial of $a + b$ has degree 4.