Solutions Single Choice 5

1. Let K be a field. Which of the following statements are true?

(a) Every algebraic extension L of K is a finite extension.

(b) Every finite extension L of K is an algebraic extension.

(c) The extension $\mathbb{Q}(\exp(2\pi i/1234)) : \mathbb{Q}$ is not algebraic.

(d) All of the above are false.

Solution: The answer is (b); which we have seen in the lectures. For part (a) note that for example the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ is infinite. Since $\exp(2\pi i/1234)^{1234} = 1$, the element $\exp(2\pi i/1234)$ is algebraic over $\mathbb{Q}$ with minimal polynomial

$$\Phi_{1234}(X) = \prod_n (X - \exp(2\pi i/1234)^n),$$

where the product is taken over all $1 \leqslant n \leqslant 1233$ such that $\gcd(n, 1234) = 1$.

2. How many subfields F of $\mathbb{Q}(\sqrt[4]{3}, i)$ exist such that $[F : \mathbb{Q}] = 4$?

(a) $\geqslant 3$

(b) 2

(c) 1

(d) 0

Solution: The answer is (a) and by explicit calculation we get that the fields $\mathbb{Q}(i\sqrt[4]{3})$, $\mathbb{Q}(\sqrt{3}, i)$ and $\mathbb{Q}(\sqrt[4]{3})$ all have degree 4 over $\mathbb{Q}$.

3. Which of the following fields are **not** subfields F of $\mathbb{Q}(\sqrt[4]{3}, i)$ such that $[F : \mathbb{Q}] = 2$?

(a) $\mathbb{Q}(i)$

(b) $\mathbb{Q}(i\sqrt{3})$

(c) $\mathbb{Q}(i + \sqrt{3})$

(d) $\mathbb{Q}(\sqrt{3})$

Solution: The answer is (c): we will show that the minimal polynomial of $i + \sqrt{3}$ has degree > 2, which implies that the degree of the field extension would be greater than 2 as well. Compute:

$$\begin{aligned}(\sqrt{3} + i)^2 = 2 + 2i\sqrt{3} &\iff (\sqrt{3} + i)^2 - 2 = 2i\sqrt{3} \quad (\in \mathbb{C}\backslash\mathbb{R}) \\ &\iff ((\sqrt{3} + i)^2 - 2)^2 + 12 = 0,\end{aligned}$$

so the minimal polynomial of $\sqrt{3} + i$ is $X^4 - 4X^2 + 16$, which has degree 4.

4. Which of the following statements are true?

(a) π is algebraic over $\mathbb{Q}(\pi^3)$

(b) The fields $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$ are **not** isomorphic as fields.

(c) The fields $\mathbb{Q}(\sqrt[4]{3})$ and $\mathbb{Q}(i\sqrt[4]{3})$ are isomorphic as fields.

(d) All statements above are true.

Solution: All the statements above are true. π gets annihilated by the polynomial $x^3 - \pi^3$ over $\mathbb{Q}(\pi^3)$. For part (b) see Exercise sheet 4. The field $\mathbb{Q}(\sqrt[4]{3})$ is obtained by adjoining a root of $X^4 - 3$ to $\mathbb{Q}$. The same is true for $\mathbb{Q}(i\sqrt[4]{3})$. Thus the two fields are isomorphic, with an isomorphism determined by sending $\sqrt[4]{3}$ to $i\sqrt[4]{3}$.

5. Which of the following statements are false?

(a) $[\mathbb{Q}(\sqrt{3}, \sqrt{6}) : \mathbb{Q}] = 4$

(b) $[\mathbb{Q}(\sqrt{3} + \sqrt{7}) : \mathbb{Q}(\sqrt{3})] = 4$

(c) $\mathbb{Q}(\sqrt{3} + \sqrt{7}) = \mathbb{Q}(\sqrt{3}, \sqrt{7})$

(d) $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}] = 6$

Solution: The correct answer is (b). For part (a), we have

$$\mathbb{Q}(\sqrt{3}, \sqrt{6}) = \mathbb{Q}(\sqrt{3}, \sqrt{2}\sqrt{3}) = \mathbb{Q}(\sqrt{3}, \sqrt{2}),$$

and we have seen in the lectures that the degree of this field extension is 4.

Note that $\mathbb{Q}(\sqrt{3}+\sqrt{7}) = \mathbb{Q}(\sqrt{3}, \sqrt{7})$: it is clear that $\mathbb{Q}(\sqrt{3}+\sqrt{7}) \subseteq \mathbb{Q}(\sqrt{3}, \sqrt{7})$. Moreover, $\sqrt{3} + \sqrt{7}$ has minimal polynomial of degree 4, so that $[\mathbb{Q}(\sqrt{3} + \sqrt{7}) : \mathbb{Q}] = 4$. But note that $[\mathbb{Q}(\sqrt{3}, \sqrt{7}) : \mathbb{Q}] = 4$, which means that the two fields must be equal. This implies $[\mathbb{Q}(\sqrt{3} + \sqrt{7}) : \mathbb{Q}(\sqrt{3})] = 2$.

For part (d), we can use the multiplicativity of the field degree as the field degrees $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]$ and $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}]$ are coprime:

$$[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] \cdot [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 2 \cdot 3 = 6.$$