Solutions Single Choice 6

1. Which of the fields below are a splitting field of the polynomial $X^4 - 3$ over $\mathbb{Q}$?

(a) $\mathbb{Q}(\sqrt[4]{3}, i)$
(b) $\mathbb{Q}(\sqrt[4]{3}, i\sqrt[4]{3})$
(c) $\mathbb{Q}(\sqrt[4]{3}, i\sqrt[2]{3})$
(d) All of the above.

Solution: The correct answer is (d). By Eisenstein's criterion, the polynomial $X^4 - 3$ is irreducible over $\mathbb{Q}$. It has zeros $\pm\sqrt[4]{3}$ and $\pm i\sqrt[4]{3}$. Hence (b) is correct. Compute $i = i\sqrt[4]{3}/\sqrt[4]{3} = i\sqrt[2]{3}/(\sqrt[4]{3})^2$, so that all the fields above are equal to each other.

2. Let K be a field. Which of the following statements is **false**?

(a) If K has no proper algebraic extensions, then every non-constant polynomial $f \in K[X]$ has at least one root in K.
(b) If each irreducible polynomial $f \in K[X]$ is linear, then K is algebraically closed.
(c) If K_1 and K_2 are algebraic closures of K, then K_1 and K_2 are isomorphic over K.
(d) If K contains a subfield which is algebraically closed, then K is algebraically closed as well.

Solution: The answer is (d): consider $\mathbb{C}(X)$. Then $\mathbb{C}(X)$ contains $\mathbb{C}$, which is algebraically closed, but the polynomial $Y^2 - X$ has no zeros in $\mathbb{C}(X)$. For (a) and (b) see theorem 2.21, for (c), see theorem 2.23.

3. Which field extension is normal?

(a) $\mathbb{F}_2(X) : \mathbb{F}_2(X^3)$
(b) $\mathbb{F}_5(X) : \mathbb{F}_5(X^5)$
(c) $\mathbb{Q}(\sqrt[4]{5}) : \mathbb{Q}$
(d) $\mathbb{R} : \mathbb{Q}$

Solution: Since $Y^5 - X^5 = (Y - X)^5 \in \mathbb{F}_5(X)[Y]$, we have that $\mathbb{F}_5(X)$ is a splitting field of $Y^5 - X^5$ over $\mathbb{F}_5(X^5)$. Thus $\mathbb{F}_5(X) : \mathbb{F}_5(X^5)$ is normal and (b) is correct.

The polynomial $Y^3 - X^3 \in \mathbb{F}_2(X^3)[Y]$ has no zeros and is hence irreducible. Let $a \in \mathbb{F}_2(X)$ be a zero of $Y^3 - X^3$. Let $f, g \in \mathbb{F}_2[X]$ be monomials such that $a = f/g$, and $g \neq 0$. Then $f^3 - X^3 g^3 = 0$, and it follows that the constant coefficient of f is zero. Write $f = X \cdot \tilde{f}$, for some $\tilde{f} \in \mathbb{F}_2[X]$, so that $X^3\tilde{f}^3 - X^3 g^3 = 0$ and thus $\tilde{f}^3 = g^3$. Then $\tilde{f} = g$ and so $a = X$. Hence the polynomial $Y^3 - X^3$ only has the zero X.

On the other hand we have

$$(Y - X)^3 = Y^3 - Y^2X + YX^2 - X^3 \neq Y^3 - X^3.$$

Hence the polynomial $Y^3 - X^3$ does not split into linear factors over $\mathbb{F}_2(X)[Y]$, and the field extension is not normal.

The irreducible polynomial $X^4 - 5$ has a zero in both $\mathbb{Q}(\sqrt[4]{5})$ and $\mathbb{R}$, but since it has complex zeros as well, it does not split into linear factors.

4. The statement: *The field extension* $\mathbb{Q}(\sqrt{2+\sqrt{2}}) : \mathbb{Q}$ *is normal,* is...

(a) true

(b) false

Solution: This is true. To show that $\mathbb{Q}(\sqrt{2+\sqrt{2}}) : \mathbb{Q}$ is normal, we have to show that all the roots of the minimal polynomial of $\sqrt{2+\sqrt{2}}$ lie in $\mathbb{Q}(\sqrt{2+\sqrt{2}})$. The minimal polynomial is given by $(X^2-2)^2-2$, so the roots are given by

$$\begin{aligned} a_1 &= \sqrt{2+\sqrt{2}} \\ a_2 &= \sqrt{2-\sqrt{2}} \\ a_3 &= -\sqrt{2+\sqrt{2}} \\ a_4 &= -\sqrt{2-\sqrt{2}}. \end{aligned}$$

Now note that $a_1^2 - 2 = \sqrt{2}$ and $a_1 a_2 = \sqrt{2}$, so that we can obtain a_2 from a_1. This proves the statement.

5. Over which field is the polynomial $X^3 + 1$ separable?

(a) $\mathbb{Q}$

(b) $\mathbb{R}$

(c) $\mathbb{F}_5$

(d) All of the above.

Solution: The correct answer is (d).

Over $\mathbb{Q}$, this polynomial is separable since it has complex zeros $-1, \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}$. Similarly over $\mathbb{R}$.

Checking all elements in $\mathbb{F}_5$, we see that the polynomial $X^3 + 1$ has no zeros in $\mathbb{F}_5$. Hence it is irreducible, as the polynomial has degree 3 (if it was reducible, there would be at least one linear term). Since the derivative does not vanish, we obtain that the polynomial is separable over $\mathbb{F}_5$.