Solutions Single Choice 8

1. Which of the following statements is true for the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q})$?

(a) $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$

(b) $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(c) $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}) \cong S_4$

(d) None of the above.

Solution: The field extension is a splitting field of the polynomial $f(X) = (X^2 - 3)(X^2 + 1)$ over $\mathbb{Q}$. The Galois group permutes the roots of f, so if $\sigma \in \mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q})$, then $\sigma(\sqrt{3}) = \pm\sqrt{3}$ and $\sigma(i) = \pm i$. Note that $\{1, i, \sqrt{3}, i\sqrt{3}\}$ form a basis of $\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}$, so there are 4 possibilities for $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q})$:

$$\sigma_{j,k} : \sqrt{3} \mapsto (-1)^j \sqrt{3}$$
$$i \mapsto (-1)^k i,$$

for $j, k \in \{0, 1\}$. Note that $\sigma_{j,k}^2 = 1$, for all j, k, so that $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2. Let $H \leqslant \mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q})$ be a subgroup of order 2, i.e. $|H| = 2$. Then the fixed field $\mathbb{Q}(\sqrt{3}, i)^H$ is only given by...

(a) $\mathbb{Q}(i)$.

(b) the fields $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{3})$.

(c) the fields $\mathbb{Q}(i), \mathbb{Q}(i\sqrt{3})$ and $\mathbb{Q}(\sqrt{3})$.

(d) the fields $\mathbb{Q}, \mathbb{Q}(i), \mathbb{Q}(i\sqrt{3})$ and $\mathbb{Q}(\sqrt{3})$.

Solution: The correct answer is (c). From the solution of Exercise 1, we know that there exist 4 elements in $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, i) : \mathbb{Q})$:

$$\sigma_{j,k} : \sqrt{3} \mapsto (-1)^j \sqrt{3}$$
$$i \mapsto (-1)^k i,$$

for $j, k \in \{0, 1\}$, where 3 of them have the order 2, so each of those generate a group of order 2. We can compute

$$\mathbb{Q}(\sqrt{3}, i)^{\langle \sigma_{1,0} \rangle} = \{a \in \mathbb{Q}(\sqrt{3}, i) : \sigma_{1,0} = a\} \supset \mathbb{Q}(\sqrt{i}),$$

but also $\mathbb{Q}(\sqrt{3}, i)^{\langle \sigma_{1,0} \rangle} \neq \mathbb{Q}(\sqrt{3}, i)$, as $\sigma_{1,0}(\sqrt{3}) = -\sqrt{3}$, and by considerng the degrees of the extensions, we get $\mathbb{Q}(\sqrt{3}, i)^{\langle \sigma_{1,0} \rangle} = \mathbb{Q}(\sqrt{3})$. Similarly we can argue in the other cases.

3. Between which field extensions does there exist a field homomorphism over $\mathbb{Q}$?

(a) $\mathbb{Q}(i) \to \mathbb{Q}(\pi)$

(b) $\mathbb{Q}(\sqrt[3]{9}) \to \mathbb{Q}(\sqrt[3]{3})$

(c) $\mathbb{Q}(i) \to \mathbb{Q}(\sqrt{3})$

(d) $\mathbb{Q}(\sqrt{3}) \to \mathbb{Q}(\sqrt[3]{3})$

Solution: The correct answer is (b). The field $\mathbb{Q}(\sqrt[3]{9})$ is contained in $\mathbb{Q}(\sqrt[3]{3})$, so we can take the inclusion. For (a), note that i is algebraic over $\mathbb{Q}$, while π is transcendental. For part (c), note that there is no homomorphism $\mathbb{Q}(i) \to \mathbb{R}$, and since $\mathbb{Q}(\sqrt{3}) \subset \mathbb{R}$, this cannot exist as well. Finally, for part (d) note that $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$, while $[\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}] = 3$, and since these are coprime there cannot exist a homomorphism between them.

4. The order of the Galois group $\mathrm{Gal}(\mathbb{F}_{27} : \mathbb{F}_3)$ is

(a) 1

(b) 2

(c) 3

(d) 4

Solution: The correct answer is (c). From the lectures we know that $\mathrm{Gal}(\mathbb{F}_{27} : \mathbb{F}_3) \cong \mathbb{Z}/3\mathbb{Z}$.

5. Which of the following statements are **false**?

(a) Let $L : M : K$ be field extensions. Then $\mathrm{Gal}(L : M) \leqslant \mathrm{Gal}(L : K)$.

(b) Let L be a splitting field of a polynomial over K. Then $|\,\mathrm{Gal}(L : K)| = [L : K]$.

(c) For $L : K$ a finite field extension, there exists $n \geqslant 1$ and an embedding $\mathrm{Aut}_K(L) \hookrightarrow S_n$.

(d) $\mathrm{Gal}(\mathbb{F}_{2^2} : \mathbb{F}_2)$ is generated by the homomorphism $\mathbb{F}_{2^2} \to \mathbb{F}_{2^2}, x \mapsto x^2$.

Solution: Part (b) is true only if f is separable; see Theorem 3.5 from the lectures. Further, since $|\,\mathrm{Aut}_K(L)| \leqslant [L : K]$, the group $\mathrm{Aut}_K(L)$ is finite, so there exists an embedding as in part (c). For part (d) see Cor. 3.7.