Solutions Single Choice 9

1. The order of the Galois group $|\operatorname{Gal}(\mathbb{Q}(\sqrt{7}, \sqrt{11}) : \mathbb{Q})|$ is equal to

(a) 2

(b) 4

(c) 6

(d) 8

Solution: The correct answer is (b). The field $\mathbb{Q}(\sqrt{7}, \sqrt{11})$ is the splitting field of the polynomial $f(x) := (x^2 - 7)(x^2 - 11)$ over $\mathbb{Q}$. The Galois group permutes the roots of f, so if $\sigma \in \operatorname{Gal}(\mathbb{Q}(\sqrt{7}, \sqrt{11}$ there are 4 possibilities for σ:

$$\begin{aligned} \sigma_{j,k} : \sqrt{7} &\mapsto (-1)^j \sqrt{7} \\ \sqrt{11} &\mapsto (-1)^k \sqrt{11}, \end{aligned}$$

for $j, k \in \{0, 1\}$. Note that $\sigma_{j,k}^2 = 1$, for all j, k, so that $\operatorname{Gal}(\mathbb{Q}(\sqrt{7}, \sqrt{11}) : \mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

2. Let $\alpha := \sqrt{7} + \sqrt{11}$. Let $\alpha_0, \ldots, \alpha_r$ be the images of α under all the automorphisms of $\operatorname{Aut}(\mathbb{Q}(\sqrt{7}, \sqrt{11}) : \mathbb{Q})$. Then the product $\alpha_0 \cdots \alpha_r$ is equal to

(a) -4

(b) $18 + \sqrt{7 \cdot 11}$

(c) $18 - \sqrt{7 \cdot 11}$

(d) 16

Solution: The correct answer is (d). Using the automorphisms of $\operatorname{Gal}(\mathbb{Q}(\sqrt{7}, \sqrt{11}) : \mathbb{Q})$ we determined above, we get

$$\begin{aligned} \alpha_0 &:= \sigma_{0,0}(\alpha) = \sqrt{7} + \sqrt{11} \\ \alpha_1 &:= \sigma_{1,0}(\alpha) = -\sqrt{7} + \sqrt{11} \\ \alpha_2 &:= \sigma_{0,1}(\alpha) = \sqrt{7} - \sqrt{11} \\ \alpha_3 &:= \sigma_{1,1}(\alpha) = -\sqrt{7} - \sqrt{11} \end{aligned}$$

Hence $\alpha_0 \cdots \alpha_3 = 16$.

3. Which of the following fields is not normal over $\mathbb{Q}$?

(a) $\mathbb{Q}(\sqrt{11}, \sqrt{13})$

(b) $\mathbb{Q}(e^{2\pi i/11})$

(c) $\mathbb{Q}(2^{1/11})$

(d) $\mathbb{Q}(\sqrt{11+\sqrt{13}}, \sqrt{11-\sqrt{13}})$

Solution: The correct answer is (c). The field $\mathbb{Q}(\sqrt{11}, \sqrt{13})$ is the splitting field field of the polynomial $(x^2-11)(x^2-13)$ over $\mathbb{Q}$, and since $\pm\sqrt{11}, \pm\sqrt{13} \in \mathbb{Q}(\sqrt{11}, \sqrt{13})$, it is normal.

The field $\mathbb{Q}(e^{2\pi i/11})$ is the splitting field of the polynomial $x^{11}-1$ over $\mathbb{Q}$. Since all the zeros of this polynomial are powers of $e^{2\pi i/11}$, and thus contained in $\mathbb{Q}(e^{2\pi i/11})$, it is normal.

The element $2^{1/11}$ is a zero of the polynomial $x^{11}-2$, which is irreducible by Eisenstein. Note that the none of the complex roots of the polynomial $x^1 1-2$ are contained in the real field $\mathbb{Q}(2^{1/11})$. Hence it is not normal.

The elements $\sqrt{11+\sqrt{13}}, \sqrt{11-\sqrt{13}}$ are zeros of the polynomial $f(x) = (x^2-11)^2-13$. Since the other two zeros of $f(x)$ are given by $-\sqrt{11+\sqrt{13}}, -\sqrt{11-\sqrt{13}}$, we obtain that $\mathbb{Q}(\sqrt{11+\sqrt{13}}, \sqrt{11-\sqrt{13}})$ is normal.

4. The subgroup $H \leqslant \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q})$ for which

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})^H = \mathbb{Q}(\sqrt{30})$$

is isomorphic to

(a) $\mathbb{Z}/2\mathbb{Z}$

(b) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(c) $\mathbb{Z}/4\mathbb{Z}$

(d) $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

Solution: The correct answer is (b). The field $L := \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ is the splitting field of the polynomial $f(x) := (x^2-2)(x^2-3)(x^2-5)$ over $\mathbb{Q}$. The Galois group $G := \mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}) : \mathbb{Q})$ permutes the zeros of f, so that for $\sigma \in G$ we have

$$\sigma(\sqrt{2}) = \pm\sqrt{2}$$
$$\sigma(\sqrt{3}) = \pm\sqrt{3}$$
$$\sigma(\sqrt{5}) = \pm\sqrt{5}$$

so that the group G has 8 automorphism of order 2, determined by the images of $\sqrt{2}, \sqrt{3}$ and $\sqrt{5}$. Hence $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Further, note that $[\mathbb{Q}(\sqrt{30}) : \mathbb{Q}] = 2$, as $\sqrt{30}$ is a zero of the degree 2 polynomial x^2-30. Then from Corollary 3.12 we have

$$|H| = \frac{[L : \mathbb{Q}]}{[L^H : \mathbb{Q}]} = \frac{8}{2} = 4.$$

The only subgroup of order 4 in G is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

5. Let $L : K$ be a finite Galois extension, such that $G = \mathrm{Gal}(L : K) \cong \mathbb{Z}/6\mathbb{Z}$. Let H be a subgroup of G of order 2, and Q be a subgroup of G of order 3. Which of the following statements is false?

(a) $[L^H : K] = \dfrac{[L : K]}{3}$

(b) $[L : K] = 6$

(c) $L^H \neq L^Q$

(d) $L : K$ is separable.

Solution: The correct answer is (a). This is false; by Corollary 3.12 we would have

$$[L^H : K] = \frac{[L : K]}{|H|} = \frac{[L : K]}{2}.$$

We have seen in the lectures that $\mathrm{Gal}(L : K) = [L : K]$ for finite Galois extensions, so (b) follows. Part (c) follows from Corollary 3.11. Part (d) follows from Theorem 4.2.