Example 3 $f(x) = (x^2-3)(x^2+1) \in \mathbb{Q}[x]$

has $K = \mathbb{Q}(\sqrt{3}, i)$ as a splitting field.

$$\begin{tikzcd}[column sep=small] & \mathbb{Q}(\sqrt{3}, i) \arrow[dl, dash, "2"'] \arrow[dr, dash] & \\ \mathbb{Q}(\sqrt{3}) \arrow[dr, dash, "2"'] & & \mathbb{Q}(i) \arrow[dl, dash, "2"] \\ & \mathbb{Q} & \end{tikzcd}$$

$[K:\mathbb{Q}] \leq 4 < 4!$

In fact $[\mathbb{Q}(\sqrt{3}, i):\mathbb{Q}] = 4$

Since x^2+1 is also irreducible over $\mathbb{Q}(\sqrt{3})$

Hence $[\mathbb{Q}(\sqrt{3}, i):\mathbb{Q}(\sqrt{3})] = 2$

4 $g(x) = (x^2-2x-2)(x^3+1)$

$(x-(1+\sqrt{3}))(x-(1-\sqrt{3}))(x+1)(x-\xi_3)(x-\xi_3^2)$

$\xi_3 = e^{2\pi i/3} = \dfrac{1+i\sqrt{3}}{2}$

Hence $\mathbb{Q}(\sqrt{3}, i)$ is also a splitting field of $g(x)$.

5 Even 2 irreducible polynomials can have the same splitting field

$h(x) = x^2-3, \quad k(x) = x^2-2x-2$

Then $\mathbb{Q}(\sqrt{3})$ is a splitting field for both polys.

6. Let $f(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$.

Since $f(0) = 1 = f(1)$, f has no roots in $\mathbb{Z}_2$, hence is irreducible.

Let α be a root of f in an extension of $\mathbb{Z}_2$

Then $\mathbb{Z}_2(\alpha) = \{a + b\alpha \mid a, b \in \mathbb{Z}_2\}$

$= \{0, 1, \alpha, 1+\alpha\}$.

is a field with 4 elts.

And $f(x)$ in fact splits in $\mathbb{Z}_2(\alpha)$

Since $(x^2 + x + 1) = (x - \alpha)(x + 1 + \alpha)$

Hence $\mathbb{Z}_2(\alpha)$ is a splitting field of $f \in \mathbb{Z}_2[x]$.

<u>Rmk</u> We have seen that for a general poly $f(x) \in F[x]$ of degree n a splitting field is of degree at most $n!$ over F, and there are polynomials with splitting field degree exactly $n!$

e.g. $x^3 - 2$ has splitting field $\mathbb{Q}(\sqrt[3]{2}, i) / \mathbb{Q}$ which has degree $3! = 6$

§2.4 Algebraic closures

Splitting field of a polynomial $f(x) \in F[x]$ is an extention of F that contains all roots of $f(x)$.

Extentions of F which contain all roots of all polys over F deserve a special name

Defn 1) A field K is called algebraically closed if every polynomial with coefs in K has a root in K.

2) An extention K of F, which is algebraic over F and algebraically closed is called an algebraic closure of F, denoted by $\overline{F}$

If F is algebraically closed then F is an algebraic closure of itself ie $\overline{F} = F$

Ex 1) $\mathbb{Q}$, $\mathbb{R}$ are not alg. closed, eg x^2+1 has no zeroes in $\mathbb{R}$ or $\mathbb{Q}$.

3) $\mathbb{C}$ is alg. closed. This is fund. thm of Algebra which we proved in function theory. One can also prove this using Galois theory as we will see.

Algebraically closed fields can be characterized in different ways. We have

Thm 2.21 If K is a field, then the following conditions are equivalent

① Every non-constant polynomial $f \in K[x]$ has at least one root in K

② Every non-constant polynomial $f \in K[x]$ splits over K.

③ Every irred. polynomial $f \in K[x]$ is linear

④ K has no proper algebraic extensions.

Proof ① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ④ $\Rightarrow$ ①

Proof ① ⇒ ② By ① we may write
$f(x) = (x-\alpha)\,g(x)$ with $g(x) \in K[x]$
and proceed inductively to show that any nonconstant poly. is a product of linear factors

② ⇒ ③ If $f \in K[x]$ is irreducible then f is nonconstant. By ② is a product of linear factors. Since it is irreducible, there can be only 1 factor and hence f is linear.

③ ⇒ ④ Let L be an alg. extension of K. If $\alpha \in L$, let $f(x) \in K[x]$ be the minimal poly of α over K. Then f is irreducible and by ③ it is of the form $x-\alpha$. But then $\alpha \in K$ and hence $L = K$

④ ⇒ ① Let $f \in K[x]$ be a non-const. poly. Adjoin a root α of f to K to obtain $K(\alpha)$ as in Kronecker's thm. But then $K(\alpha)$ is an alg. extension of K, so by ④ $K(\alpha) = K$ and $\alpha \in K$.

$\blacksquare$

The next two theorems guarantees the existence of algebraic closure, which we give without proofs.

Thm 2.22 For any field F, there exists an algebraically closed field K containing F

Thm 2.23 Let K be an alg. closed field and F a subfield of K.
Then the collection of elements of K which are algebraic over F is an algebraic closure of F,
If K_1 and K_2 are 2 alg. closures of F, then K_1 and K_2 are isomorphic over F. Ie $\exists$ an isom
$\phi : K_1 \to K_2$, s.t $\phi|_F = id$ on F.

Ex ① The algebraic closure of $\mathbb{R}$ is $\mathbb{C}$ since, $\mathbb{C}$ is algebraically closed and $\mathbb{C} : \mathbb{R}$ is an algebraic extension
$\mathbb{C} = \mathbb{R}(i) = \mathbb{R}[x]/(x^2+1)$

② The algebraic closure of $\mathbb{Q}$ is
$\overline{\mathbb{Q}}$ = set of all $\alpha \in \mathbb{C}$, which is algebraic over $\mathbb{Q}$.

Note $\mathbb{C} \neq \overline{\mathbb{Q}}$, ie $\mathbb{C}$ is not an alg closure of $\mathbb{Q}$. It contains transcendental elements as e, π.