When the extension is not normal, we can either look at alg closure $\bar{L}$ of L which will have all the roots or go to an extension N of L which is normal (and smaller than $\bar{L}$) ($N \subset \bar{L}$)

Defn Let $L:K$ be an algebraic extension. A normal closure of $L:K$ is an extension N of L s.t

1) $N:K$ is normal
2) If $L \subseteq M \subseteq N$ and $M:K$ is normal then $M = N$.

ie N is the smallest extension of L which is normal over K.

We have that normal closures exist and unique

Thm 4.6 If $L:K$ is a finite extension then there exists a normal closure N of $L:K$ which is a finite extension of K.
If M is another normal closure then the extension $M:K$ and $N:K$ are isomorphic

Proof Exercise: Hint: let $\alpha_1, \dots \alpha_n$ be basis of L over K, let m_i be the minimal poly of α_i over K
let N be the splitting field of $m_1 \dots m_n$ over K

Example $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$ not normal. Its normal closure is $\mathbb{Q}(\sqrt[3]{2}, \rho)$

We obtain a normal closure by adjoining "missing" roots.

The next lemma shows that for a finite extension $K:L$, the image of any K-monom $L \to M$ lands in the normal closure (where $N \subseteq M$)

Lemma 4.7 Suppose $K \subseteq L \subseteq N \subseteq M$ where $L:K$ finite, N is a normal closure of $L:K$. If ϕ is a K-monom $L \to M$ then $\phi(L) \subseteq N$

Proof Let $\alpha \in L$, $m_{\alpha, K}$ its min poly / K.

$$0 = m(\alpha) = \phi(m(\alpha)) = m(\phi(\alpha))$$

So that $\phi(\alpha)$ is a zero of m, hence lies in N since $N:K$ is normal

Hence $\phi(L) \subseteq N$ ☒

Rmk. If $K \subset L$ is normal and we have $K \subset L \subset M$ then any K-mon $\phi : L \to M$ is in fact a K-autom of L since $\phi(L) \subseteq L = N$ using Lemma 4.7. But ϕ is a K-lin. map of a finite dim'l vector spaces which is injective hence it is also surjective, hence $\phi(L) = L$, Hence ϕ is a K-autom of L

In fact
The next thm gives a description of a normal extention in terms K-monomorphisms of $L \to M$.

Thm 4.8 Let $L:K$ be a finite extension. Then the following are equivalent

1. $L:K$ is normal
2. For every extension M of K containing L, every K-monom $\phi: L \to M$ is a K-autom of L
3. $\exists$ a normal extension N of K containing L s.t every K-monom $\phi: L \to N$ is a K-autom of L

Proof Exercise $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

remark above.

The following thm is about separable extensions

Thm 4.9 Suppose $L:K$ is finite and separable of degree n. Then there are exactly n K-monom of L into a normal closure N.

Proof Exercise

As a corollary we get another proof of

Thm 4.10 $L:K$ finite sep, normal $\Rightarrow$ $L:K$ Galois

Proof

By thm 4.9, $\exists$ exactly n distinct K mon of L into $N = L$

Since L is normal, these n-distinct K mon. are actually K-autom of L (by thm 4.8).

Hence $|Gal(L:K)| \geq n$. Since

$$|Gal(L:K)| \leq [L:K] = n$$

we have $|Gal(L:K)| = [L:K]$

hence $L:K$ is Galois

§5. The Galois correspondence.

We can now give the fundamental result on the Galois correspondence for a normal seperable finite extension (ie Galois extension) $L:K$.

Thm 5.1 (The Fundamental Thm of Galois theory).

Let $L:K$ be a finite normal, seperable extension of degree n with $G = Gal(L:K)$.

Let $\mathcal{F} := \{ M \mid L \supset M \supset K \}$ be the set of subfields of L containing K.

and $\mathcal{G} := \{ H \mid H \leq G \}$ the set of subgroups of G

with the 2 maps
$$\gamma : \mathcal{F} \longrightarrow \mathcal{G}$$
$$M \longmapsto Gal(L:M)$$

and
$$\phi : \mathcal{G} \longrightarrow \mathcal{F}$$
$$H \longrightarrow L^H = Fix\, H = \{ l \in L \mid \sigma(l) = l \ \forall \sigma \in H \}$$

Then

① $|G| = |\mathrm{Gal}(L:K)| = [L:K] = n$

② The maps γ, ϕ are mutual inverses and set up an order reversing one-to-one correspondence between the sets $\mathcal{F}$ and $\mathcal{G}$.

③ If M is an intermediate field, then $|\gamma(M)| = |\mathrm{Gal}(L:M)| = [L:M]$ ie $L:M$ is a Galois extension

$$[M:K] = [L:K]/[L:M] = |G|/|\gamma(M)|$$

④ An intermediate field M is a normal extension of $K \Leftrightarrow \gamma(M) \trianglelefteq G$.

ie $\gamma(M)$ is a normal s/gp

⑤ If an intermediate field M is normal over K then $\mathrm{Gal}(M:K) \cong G/\gamma(M)$

Before we give the proof, let's look at a simple example.

Example: $f(x) = (x^2+1)(x^2-5) \in \mathbb{Q}[x]$

$L = \mathbb{Q}(i, \sqrt{5})$ is a splitting field of sep. pol f

$[L:K] = |\mathrm{Gal}(L:K)| = 4$

Any $\mathbb{Q}$-autom of L must send $i \to \pm i$
$\sqrt{5} \to \pm\sqrt{5}$

call $a = i$, $b = -i$
$c = \sqrt{5}$ $d = -\sqrt{5}$

Then the 4 possible $\mathbb{Q}$-autom of L are

$e = $ identity

$\sigma_1 = (a\ b)$ ie $\sigma_1 : i \to -i$, $\sqrt{5} \to \sqrt{5}$

$\sigma_2 = (c\,d)$ $\sigma_2 : i \to i$, $\sqrt{5} \to -\sqrt{5}$

$\sigma_3 = (a\,b)(c\,d)$ $\sigma_3 : i \to -i$, $\sqrt{5} \to -\sqrt{5}$

$G = \{e, \sigma_1, \sigma_2, \sigma_3\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$$\begin{tikzcd}[column sep=small]
 & G = \mathrm{Gal}(L:\mathbb{Q}) = \{e, \sigma_1, \sigma_2, \sigma_3\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
H_1 = \{e, \sigma_1\} \arrow[dr, dash] & H_2 = \{e, \sigma_2\} \arrow[d, dash] & H_3 = \{e, \sigma_3\} \arrow[dl, dash] \\
 & \{e\} &
\end{tikzcd}$$

G has 3 proper subgroups of order 2

$\text{Fix}\, H_1 = \text{Fix}\{e, (ab)\} = \mathbb{Q}(\sqrt{5})$. To see this note since $\text{Fix}\, H_1$ fixes $c = \sqrt{5}$ and $d = -\sqrt{5}$, $\mathbb{Q}(\sqrt{5}) \subset \text{Fix}\, H_1$, Hence $[\text{Fix}\, H_1 : \mathbb{Q}] \geq 2$.
Since $\text{Fix}\, H_1 \subsetneq L$ (it does not fix i), $[\text{Fix}\, H_1 : \mathbb{Q}] < 4$.
Therefore $[\text{Fix}\, H_1 : \mathbb{Q}] = 2$ and $\text{Fix}\, H_1 = \mathbb{Q}(\sqrt{5})$.
Similarly, $\text{Fix}\, H_2 = \text{Fix}\{e, (cd)\} = \mathbb{Q}(i)$
$\text{Fix}\, H_3 = \text{Fix}\{e, (ab)(cd)\} = \mathbb{Q}(i\sqrt{5})$

$\text{Fix}\{e\} = L$ $\qquad$ $\text{Fix}(G) = \mathbb{Q}$.

$$\begin{tikzcd}
 & G = \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[dl, equal] \arrow[d, equal] \arrow[dr, equal] & \\
H_1 \cong \mathbb{Z}_2 \arrow[dr, equal] & H_2 \cong \mathbb{Z}_2 \arrow[d, equal] & H_3 \cong \mathbb{Z}_2 \arrow[dl, equal] \\
 & \{e\} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{Fix}\, G = \mathbb{Q} \arrow[dl, equal] \arrow[d, equal] \arrow[dr, equal] & \\
\mathbb{Q}(\sqrt{5}) \arrow[dr, equal] & \mathbb{Q}(i) \arrow[d, equal] & \mathbb{Q}(i\sqrt{5}) \arrow[dl, equal] \\
 & \text{Fix}\, e = L = \mathbb{Q}(\sqrt{5}, i) &
\end{tikzcd}$$

Note in this case, - all extensions $\mathbb{Q}(\sqrt{5}) : \mathbb{Q}$, $\mathbb{Q}(i) : \mathbb{Q}$, $\mathbb{Q}(i\sqrt{5}) : \mathbb{Q}$ are all normal, with each Galois gp isomorphic to $G/\mathbb{Z}_2 \cong \mathbb{Z}_2$