Abelian groups

We have seen that every abelian group can be viewed as a $\mathbb{Z}$-module

A non-abelian group is not a $\mathbb{Z}$-module since modules always have commutative addition by definition.

The main structure theorem on finitely generated abelian groups is a special case of a general thm on finitely generated modules over PIDs.

Many concepts in abelian groups have counterparts in modules over rings

Homomorphism of groups $\longleftrightarrow$ R-lin. maps
R-module homs.

Subgroup $\longleftrightarrow$ R-submodule

Cyclic group $G = \langle g \rangle = \{ng \mid n \in \mathbb{Z}\}$ $\longleftrightarrow$ cyclic R-module $M = Rm$ for some $m \in M$.

G finitely generated $= \mathbb{Z}g_1 + \dots \mathbb{Z}g_k$ $\langle g_1, \dots g_k \rangle$ $\longleftrightarrow$ fin. gen. R module $M = Rm_1 + \dots + Rm_k$

Defn An element $m \in M$, M an R-module (R ID) is called a torsion element if $rm = 0$ for an $r \neq 0$ in R.

Torsion submodule of M is the set of all torsion elements
ie $\mathrm{Tor}(M) := \{m \in M \mid rm = 0 \text{ for some } r \neq 0\}$.

M is called a torsion module if all elements of M are torsion elements.

Rmk These concepts are counterparts of elts of finite order, torsion subgroup and torsion group in abelian groups.

Similarly M is called a torsion free module if M has no torsion element besides 0

Rmk
① Note in the definition of torsion elements we require R to be an integral domain so that the set of torsion elements become a submodule.

eg $R = \mathbb{Z}/6\mathbb{Z}$ and we consider R as an R-mod
then torsion elements of R are $\{0, 2, 3, 4\}$
It is not closed under addition $2 + 3 = 5$
and 5 is not torsion

Note $\mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$-module is a torsion module since every elt is torsion.

(2) An ideal I in R is a cyclic R-module means that it is principal.

(3) For an ideal I, R/I is a cyclic R-module since any $r+I \in R/I$ is equal to $r(1+I)$. Hence $1+I$ generate R/I

(4) For a non-zero ideal I in an ID R
R/I is a torsion R-module.
Since for all $m = r+I \in R/I$, take $a \in I \setminus \{0\} \subset R$
then

$$am = ar + I = I = 0_{R/I}$$

since $ar \in I$.

eg $\mathbb{Z}/n\mathbb{Z}$ is a torsion $\mathbb{Z}$-module
ie a torsion abelian group.

(5) $G = \mathbb{C}^\times = \mathbb{C}\setminus\{0\}$ is a multiplicatively written abelian group.
Each $n \in \mathbb{Z}$ acts on $z \in \mathbb{C}^\times$ by $n.z = z^n$
Its torsion subgroup $\mu(\mathbb{C}) := \{z \in \mathbb{C} \mid z^n = 1$ for some $n\}$
is the group of roots of unity.
Hence torsion s/gps can be infinite.

Modules over PID's

In a vector space V over a field F, if V has a finite spanning set then it has a finite basis and it is isomorphic to F^n for some n.

If we replace F with a comm ring R it is no longer true that every R-module M with a finite generating set has a basis
Not all modules have a basis.
eg. $(2, 1+\sqrt{5}) = I$ as a $\mathbb{Z}[\sqrt{-5}]$-module is fin. generated but has no basis.

In a vector space with finite basis, a subspace also has a finite basis
the example above shows this is not true either $M = \mathbb{Z}[\sqrt{-5}]$ as a $\mathbb{Z}[\sqrt{-5}]$ module has basis $\{1\}$ but I is a submodule with no basis

It turns out that when the ring R is a PID we recover a lot of the nice properties
The following theorem is very important and shows that if N is a submodule of a finitely gen. module m of rank n over a PID then N is also a free module of finite rank and it is possible to choose generators of M and N which are related

More precisely one has

Thm 9.6 Let R be a PID, M a free R-module of rank n and let N be a submodule of M. Then

① N is free of rank m, $m \leq n$.

② there exists a basis $y_1, \ldots y_n$ of M so that

and $a y_1, a y_2, \ldots, a y_m$ is a basis of N

where $a_1, a_2, \ldots a_m$ are non-zero elements of R

with divisibility relations

$$a_1 \mid a_2 \cdots \mid a_m$$

An R-module M is called cyclic if there is an element $x \in M$ s.t $M = Rx$. We can consider the R-module hom $\pi : R \to M$ which is surjective
$r \to rx$

since $M = Rx$ and we get using isom thm
$R/\ker\pi \cong M$. If R is a PID then $\ker\pi = (a)$ for some a and $\mathrm{Ann}(M) = \{r \in R \mid rm = 0 \ \forall m \in M\}$
$= (a)$, an ideal of R.

Cyclic modules are the simplest modules and the fund thm on finitely generated modules over PID say any module is isom to direct sum of such modules and we have

Thm 9.7 (Fundamental thm on f.g. modules over PID) (Invariant factor form)

Let R be a PID, M finitely generated R-module. Then

① M is isomorphic to the direct sum of finitely many cyclic modules.

More precisely

$$M \cong R^r \oplus R/(a_1) \oplus R/(a_2) \oplus \cdots \oplus R/(a_m)$$

for some integer $r \geq 0$ and non-zero elts $a_1, \ldots a_m \in R$ which are not units in R and satisfy the divisibility criteria

$$a_1 \mid a_2 \cdots \mid a_m$$ (a_i's are called invariant factors of M)

(r is called the free rank of M)

② M is torsion free if and only if M is free

③ $\operatorname{Tor}(M) \cong R/(a_1) \oplus \cdots \oplus R/(a_m)$

In particular M is torsion module if and only if $r=0$

In this case, $\underbrace{\operatorname{Ann}(M)}_{\text{annihilator of } M} := \{ r \in R \mid rm = 0 \ \forall m \in M \}$
$= (a_m)$

Rmk The proof idea for the fund thm is as follows

let $x_1, \dots x_n$ be a set of generators of M of minimal cardinality., let R^n be the free R-module of rank n with basis $b_1, \dots b_n$.

Define the hom $\pi : R^n \to M$
$$b_i \mapsto x_i \quad \forall i$$

we have $R^n / \ker\pi \cong M$

Now apply thm 9.5 with R^n and $\ker\pi$ as the submodule. Then we can choose another basis $y_1, \dots y_n$ of R^n so that $a_1y_1, \dots a_my_m$ is a basis of $\ker\pi$ for some elements $a_1, \dots, a_m \in R$ with $a_1 | a_2 \dots | a_m$

Hence $M \cong R^n / \ker\pi$
$$\cong (Ry_1 \oplus \dots \oplus Ry_n)/(Ra_1y_1 \oplus \dots \oplus Ra_my_m)$$

To see that the quotient on the right is isom to $R/(a_1) \oplus \dots \oplus R/(a_m) \oplus R^{n-m}$

Consider the natural surjection hom
$$\phi : Ry_1 \oplus \dots \oplus Ry_n \longrightarrow R/(a_1) \oplus \dots \oplus R/(a_m) \oplus R^{n-m}$$
$$(\alpha_1 y_1, \dots \alpha_n y_n) \longmapsto (\alpha_1 \bmod (a_1), \dots \alpha_m \bmod (a_m), \alpha_{m+1}, \dots, \alpha_n)$$

check that $\ker\phi \cong Ra_1y_1 \oplus \dots \oplus Ra_my_m$

(Note $\alpha_1 y_1 \dots \alpha_n y_n \in \ker\phi \Rightarrow a_i | \alpha_i \quad i = 1, \dots, m$)

Since $R/(a)$ is a torsion R-module for $a \neq 0$, M is torsion free $\Leftrightarrow M \cong R^r$.

Since R is a PID, it is also a UFD and any $a \in R$ can be written as
$a = u\, p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ $u \in R^\times$, p_i's primes

Using Chinese remainder theorem we have that
$$R/(a) \cong R/(p_1^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots \oplus R/(p_s^{\alpha_s})$$

as rings and as R-modules.
Hence the fund thm can also be written as

Thm 9.8 (Fund thm - elementary divisors version)
R a PID, M f.g. R-module then

$$M \cong R^r \oplus R/(p_1^{a_1}) \oplus \cdots \oplus R/(p_t^{a_t})$$

The prime power $p_i^{a_i}$ are uniquely determined and called elementary divisors of M.

Rmk Applying the fund thm on f.g modules over PID's to the case of f.g. $\mathbb{Z}$-modules we obtain the fund. thm on f.g. abelian groups.

One also has the uniqueness statement in the fund thm

Thm 9.9 (Fundamental thm, uniqueness)
Let R be a PID

① Two finitely generated R-modules M_1 and M_2 are isomorphic if and only if they have the same free rank and the same list of invariant factors

② Two finitely generated R modules M_1 and M_2 are isomorphic iff they have the same rank and the same list of elementary divisors.

Rmk ① The elementary divisors of M are the prime power factors of the invariant factors of M

② The largest inv. factor of M is the product of the largest of the distinct prime powers among the elementary divisors of M.
The next largest inv. factor is the product of the largest of the distinct prime powers among the remaining elementary divisors of M and etc.