Problem sheet 1

Problem 1

Let A be 2×2 real matrix with eigenvalues $\lambda \in (1, \infty)$ and $\mu \in (0, 1)$. Consider the map

$$T : S^1 \to S^1 : x \mapsto \frac{Ax}{\|Ax\|}.$$

(a) Show that T has exactly four fixed points.

(b) Show that $T^n x = \frac{A^n x}{\|A^n x\|}$.

(c) Show that for every $x \in S^1$, the sequence $T^n x$ converges to a fixed point of T.

Problem 2

Let R_α, R_β: $S^1 \to S^1$ be circle rotation by angles α and β respectively. Consider the map

$$T : S^1 \times S^1 \mapsto S^1 \times S^1 : (x, y) \mapsto (R_\alpha x, R_\beta y).$$

(a) Show that for any $(x, y) \in S^1 \times S^1$ there exists $n_i \to \infty$ such that

$$\lim_{i \to \infty} T^{n_i}(x, y) = (x, y).$$

(b) Show that when $\alpha = 2\beta$, T is not transitive.

Problem 3

Let $T : S^1 \to S^1$ be the doubling map $x \mapsto 2x \bmod 1$.

(a) For $n \in \mathbb{N}$ find all periodic points x of period n by solving directly the equation $T^n(x) = x$.

(b) Show that the periodic points for the doubling map are dense in S^1.

Problem 4

Given any finite sequence of decimal digits $\omega_1 \cdots \omega_l$, show that there exists $n \in \mathbb{N}$ such that the decimal expansion 2^n starts with $\omega_1 \cdots \omega_l$. (Hint: Use that $\log_{10} 2$ is irrational.)

Problem 5

Let $k \geq 2$ and

$$\Omega = \prod_{i=1}^{\infty} \{0, ..., k-1\}$$

with the metric $d(\omega, \eta) = \sum_{i=1}^{\infty} \frac{|\omega_i - \eta_i|}{k^i}$. Consider the shift map

$$S : \Omega \to \Omega : (\omega_i)_{i \geq 1} \to (\omega_{i+1})_{i \geq 1}.$$

(a) Show that $\omega^{(n)} \to \omega$ as $n \to \infty$ in Ω if and only if $\omega_i^{(n)} \to \omega_i$ as $n \to \infty$ for all i.

(b) How many S-periodic points of period n?

(c) Show that the set of S-periodic points is dense in Ω.