Problem sheet 10

Problem 1

Prove that the hyperbolic Riemannian metric given by $\langle v, w\rangle_z \colon = \frac{1}{y^2}(v \cdot w)$, for $z = x + i \cdot y \in \mathbb{H}$, and $v, w \in \mathrm{T}_z\mathbb{H}$ is indeed a metric, and that it induces the same topology as does the Euclidean norm $\|\cdot\|_2$ on $\mathbb{C} \supseteq \mathbb{H}$.

Problem 2

Recall the definition of the geodesic flow on the Hyperbolic plane. Using the same notation as seen in class show the following:
Prove that if $(z, \mathbf{v})$ has the property that the distance between the base point of $g_t((i,i))$ and that of $g_t((z,v))$ converges to zero as $t \to \infty$, then the imaginary part of z is equal to 1, and $\mathbf{v} = i$.

Problem 3

Let $X = \{0, \frac{1}{n} \mid n \geq 1\}$ with the compact topology inherited from the reals. Since X is countable, there is a bijection $\theta \colon X \to \mathbb{Z}$ – we fix an arbitrary such θ.
Show that the map $T \colon X \to X$ defined by $T(x) = \theta^{-1}(\theta(x)+1)$ is measurable with respect to the Borel σ-algebra on X but has no invariant probability Why does this not contradict the theorem on existence of invariant measures proven in the lecture?

Problem 4

Let $X = \mathbb{T}^3$. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

1. Construct a map $T \colon X \to X$, a function $f \colon X \to \mathbb{R}$ and a point $x_0 \in X$ such that
$$f(T^n x_0) = \{n^3\alpha\} \quad \text{for all } n \geq 1.$$
Here $\{\cdot\}$ denotes the fractional part.
Hint: similar as in Furstenberg's Theorem

2. Let $p(x)$ be any polynomial of degree 3.
Construct a map $T \colon X \to X$, a function $f \colon X \to \mathbb{R}$ and a point $x_0 \in X$ such that
$$f(T^n x_0) = \{p(n)\alpha\} \quad \text{for all } n \geq 1.$$

Problem 5

Show that a surjective homomorphism $T\colon X \to X$ of a compact group X – that is a group X which is equipped with a topology, such that X seen as a topological space is compact – is uniquely ergodic if and only if $|X| = 1$.