Problem sheet 11

Problem 1

In this exercise we finish the proof of the fundamental domain of the $\mathrm{PSL}_2(\mathbb{Z})$ action on the upper half plane.
Let

$$\begin{aligned} E =&\{z \in \mathbb{H} \mid |z| > 1,\ |\operatorname{Re}(z)| < \tfrac{1}{2}\} \\ &\cup\{z \in \mathbb{H} \mid |z| > 1,\ \operatorname{Re}(z) = -\tfrac{1}{2}\} \\ &\cup\{z \in \mathbb{H} \mid |z| = 1,\ -\tfrac{1}{2} \leq \operatorname{Re}(z) \leq 0\} \end{aligned}$$

be given as described in the lecture. Assume that $z \in E$ and $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ is such that $\gamma \cdot z \in E$. Prove that:

- either $z = i$;
- or z is the third root of unity (i.e. $z^3 = 1$) which is included in E.

Determine in both cases what the matrix $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ could look like.

Problem 2

Show that the closed linear group

$$T = \left\{ \begin{pmatrix} e^{t/2} & s \\ & e^{-t/2} \end{pmatrix} \mid s, t \in \mathbb{R} \right\}$$

does not contain a lattice. That is, T does not contain a discrete subgroup with a fundamental domain of finite left Haar measure.

Problem 3

Prove that $\mathrm{PSL}_2(\mathbb{Z})$ is a free product (see definition of free product) of an element of order 2 and an element of order 3.
Hint: To find the two generating elements recall the definition of the fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$*. In particular, solve Problem 1 first.*

Problem 4

Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a uniform lattice and fix a point $x \in X = \Gamma\backslash \mathrm{PSL}_2(\mathbb{R})$. Define

$$U^- = \left\{ \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}.$$

Prove that xU^- consists precisely of all the points $y \in X$ for which

$$d(R_{a_t}(x), R_{a_t}(y)) \to 0, \quad \text{as } t \to \infty,$$

where $R_{a_t}\colon X \to X$ describes the right multiplication by $a_t^{-1} = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$.

Problem 5

Show that the horocycle flow $\mathrm{PSL}_2(\mathbb{Z})gU^-$ is compact if and only if $g(\infty) = \lim_{|z|\to\infty} g(z)$ lies in $\mathbb{Q} \cup \{\infty\}$.
Show that if $\mathrm{PSL}_2(\mathbb{Z})gU^-$ is compact, then any other compact orbit is of the form $\mathrm{PSL}_2(\mathbb{Z})gaU^-$ for some

$$a \in A = \left\{ \begin{pmatrix} e^{t/2} & s \\ & e^{-t/2} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$