Problem sheet 2

Problem 1

Prove that the doubling map $x \mapsto 2x \bmod 1$ is topologically mixing.

Problem 2

Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be the tripling map $x \mapsto 3x \bmod 1$. Consider the Cantor set

$$C := \left\{ \sum_{i=1}^{\infty} \frac{a_i}{3^i} : a_i \in \{0, 2\} \right\}.$$

(a) Show that $T^n(x) \in C$ for any $x \in C$ and $n \in \mathbb{N}$.

(b) Show that there exists $x \in \mathbb{T}$ with $\overline{\mathcal{O}^+(x)}$ uncountable but not $\mathbb{T}$, where $\mathcal{O}^+(x)$ is the forward orbit $\{T^n(x) : n \in \mathbb{N}\}$.

Problem 3

Let $p \geq 2$ and let $T_p : \mathbb{T}^1 \to \mathbb{T}^1$ be the $\times p$ map $x \mapsto px \bmod 1$. For $x_0 = \sum_{k=1}^{\infty} \frac{1}{p^{k!}}$ show that $\overline{\mathcal{O}^+(x_0)}$ is countable but not finite.

Problem 4

Let X_1 and X_2 be compact metric spaces and $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$. Consider the space $X = X_1 \times X_2$ with metric

$$d_X\big((x_1, x_2), (y_1, y_2)\big) := \max\big(d_{X_1}(x_1, y_1), d_{X_2}(x_2, y_2)\big)$$

and the map $T : X \to X : (x_1, x_2) \to \big(T_1(x_1), T_2(x_2)\big)$.

(a) Show that if T_1 and T_2 are topologically mixing, then T is also topologically mixing.

(b) Is (a) true for "topologically transitive"?

Problem 5

Consider the dynamical system $T : X \to X$ for a compact metric space X. For a subset $S \subseteq X$ we define the ω-limit of S as

$$\omega^+(S) = \{\lim_{k\to\infty} T^{n_k}(y_k) \mid y_k \in S, n_k \nearrow \infty\}.$$

(a) Prove that $\omega^+(S)$ is T-invariant,

(b) Prove that $\omega^+(S)$ is closed.

(c) Find an example of a dynamical system, where

$$\overline{\bigcup_{x\in S} \omega^+(x)} \neq \omega^+(S).$$

Which inclusion does always hold?

Problem 6

Let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be the map

$$T(x) := \begin{cases} 0 & \text{if } x = 0, 1, \\ nx - 1 & \text{if } x \in [\frac{1}{n}, \frac{1}{n-1}),\ n \geq 2. \end{cases}$$

(a) Show that for $x \in \mathbb{Q}$ there exists $n \in \mathbb{N}$ with $T^n(x) = 0$.
(b) Show that for $x \in \mathbb{R} \setminus \mathbb{Q}$ we have $T^n(x) \neq 0$ for all $n \in \mathbb{N}$.
(c) Show that e is irrational using this dynamical system.