Problem sheet 3

Problem 1

Let X be compact metric space and $T : X \to X$ be a continuous map. Show that for any $x \in X$ the forward omega limit $\omega^+(x)$ is a T-invariant closed set.

Problem 2

Find an example of a continuous map $T : \mathbb{T}^1 \to \mathbb{T}^1$ such that

(a) T is minimal but not topologically mixing.

(b) T is topologically mixing but not minimal.

Problem 3

Consider the circle rotation $R : \mathbb{T}^1 \to \mathbb{T}^1 : x \mapsto (x + \alpha) \pmod 1$ with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $h : \mathbb{T}^1 \to \mathbb{T}^1$ be a homeomorphism such that $h(R(x)) = R(h(x))$ for all $x \in \mathbb{T}^1$. Show that then $h(x) = (x + \beta) \pmod 1$ for all $x \in \mathbb{T}^1$ for some $\beta \in \mathbb{R}$. (Hint: use that $\{n\alpha \pmod 1\}_{n\geq 1}$ is dense in $\mathbb{T}^1$.)

Problem 4

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ for $n \geq 1$. Using the multiple recurrence theorem for the rotation maps $R_{\alpha_1}, \ldots, R_{\alpha_n}$, show that for any $\epsilon > 0$ there exist infinitely many $q \in \mathbb{N}$ and $p_1, \ldots, p_n \in \mathbb{Z}$ such that $|\alpha_k - \frac{p_k}{q}| < \frac{\epsilon}{q}$ for all $1 \leq k \leq n$.

Problem 5

Let $T : X \to X$ be a continuous map. We say that T is *topologically weak mixing* if the map

$$T \times T : X \times X \to X \times X : (x_1, x_2) \to \big(T(x_1), T(x_2)\big)$$

is topologically transitive.

(a) Prove that if T is topologically weak mixing, then T is topologically transitive.

(b) Prove that if T is topologically mixing, then T is topologically weak mixing.

Problem 6

The *Gauss map* $G : [0,1] \to [0,1]$ is defined as follows:

$$G(x) := \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} - n & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n}],\ n \geq 2. \end{cases}$$

(a) Prove that $G^n(x) = 0$ for some $n \in \mathbb{N}$ if and only x is rational.

(b) Prove that if $G^n(x) = x$ for some $n \in \mathbb{N}$ then x is quadratic irrational.