Problem 1

Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(a) Draw the corresponding graphs G_{A_1} and G_{A_2}.

(b) Investigate whether A_i's are connected and aperiodic.

(c) Investigate whether the corresponding maps $S_i : \Omega_{A_i} \to \Omega_{A_i}$ on the vertex shift are topologically transitive and topologically mixing.

Problem 2

Let A be the adjacency matrix of a finite graph and let $S : \Omega_A \to \Omega_A$ be the corresponding map on the vertex shift. Show that for any $n \in \mathbb{N}$ the cardinality of the set of $x \in \Omega_A$ with $S^n(x) = x$ is exactly the trace $\mathrm{Tr}(A^n)$.

Problem 3

Prove that the odd shift $X_{\text{odd}} \subset \{0,1\}^{\mathbb{Z}}$ is sofic but not of finite type, where X_{odd} is the set of $x \in \{0,1\}^{\mathbb{Z}}$ such that any two 1's in the sequence are separated by an odd number of 0's.

Problem 4

Let $\mathcal{W}$ denote the set of words in 0 and 1. For $w = i_1 \cdots i_l \in \mathcal{W}$, define $w' = \bar{i_1} \cdots \bar{i_l}$ (here $\bar{i}$ denotes the negation of i). Let $w_1 = 1$ and $w_{n+1} = w_n w_n'$ for $n \geq 1$ and $w_\infty \in \{0,1\}^{\mathbb{N}}$ is the infinite sequence of 0 and 1 obtained in this way. Recall that the metric d on $\{0,1\}^{\mathbb{N}}$ is defined by $d(\omega, \eta) = \sum_{i=1}^{\infty} \frac{|\omega_i - \eta_i|}{2^i}$.

(a) Define the substitution maps $s : \mathcal{W} \to \mathcal{W}$ by $s(0) = 11$, $s(1) = 10$, $s(i_1 \cdots i_l) = s(i_1) \cdots s(i_l)$. Show that $w_n = s^{n-1}(1)$ for all $n \in \mathbb{N}$.

(b) Show that the point $w_\infty \in \{0,1\}^{\mathbb{N}}$ is uniformly recurrent for the shift map, i.e. for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that there is an increasing sequence $n_i \to \infty$ such that $d(\sigma^{n_i} w_\infty, w_\infty)$ and $|n_{i+1} - n_i| < N$ for all $i \in \mathbb{N}$, where σ is the shift map on $\{0,1\}^{\mathbb{N}}$.

(c) Show that $w_\infty \in \{0,1\}^{\mathbb{N}}$ is not periodic for the shift map.

Problem 5

Let $\mathcal{W}$ denote the set of words in 0 and 1. Recall that the word metric on $\mathcal{W}$ is defined by $d(i_1 \cdots i_l, j_1 \cdots j_m) = 2^{-k}$, where k is the smallest integer such that $i_k \neq j_k$ (if $i_k = j_k$ for all $1 \leq k \leq \min(l, m)$ then $k = \min(l, m) + 1$). Note that $\{0, 1\}^{\mathbb{N}}$ can be viewed as the completion of $\mathcal{W}$.

Define the substitution maps $\zeta : \mathcal{W} \to \mathcal{W}$ by $\zeta(0) = 01$, $\zeta(1) = 0$, and $\zeta(i_1 \cdots i_l) = \zeta(i_1) \cdots \zeta(i_l)$.

(a) Show that $u = \lim_{n\to\infty} \zeta^n(0) \in \{0, 1\}^{\mathbb{N}}$ exists, i.e. $\{\zeta^n(0)\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathcal{W}$.

(b) Prove that the Fibonacci sequence $u = 01001010010\ldots$ obtained in (a) is Sturmian, i.e. for any $n \geq 1$ the number of words of length n appearing in u is $n + 1$.