Problem sheet 5

Problem 1

Let $T : [0, 1] \to [0, 1]$ be the map defined by $T(x) = x^2$.

(a) Show that for every T-invariant probability measure μ, $\mu\big((0, 1)\big) = 0$. (Hint: Poincare Recurrence Theorem)

(b) Describe all T-invariant probability measures.

Problem 2

Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) = 1$, and $T : X \to X$ be an ergodic measure-preserving map. Let $f : X \to \mathbb{R}^+$ be a measurable function such that $\int_X f d\mu = \infty$. Prove that

$$\frac{1}{N}\sum_{k=0}^{N-1} f(T^k x) \to \infty$$

for almost all $x \in X$. (Hint: approximate f by integrable functions.)

Problem 3

Let $X = [0, 1]^2$ and let $\mathcal{A}$ be the σ-algebras comprising sets of the form $B \times [0, 1]$ for B a measurable subset of $[0, 1]$. Define $g : X \to \mathbb{R}$ by $g(x, y) = xy + y^2$.

(a) Compute the conditional expectation $E(g|\mathcal{A})$ with respect to the two-dimensional Lebesgue measure.

(b) Let μ be the measure on X defined by

$$\int f d\mu = \frac{1}{3}\int_0^1 f(s, s)ds + \int_0^1 \int_0^{\sqrt{s}} f(s, t)dtds.$$

Compute the conditional expectation $E(g|\mathcal{A})$ with respect to μ.

Problem 4

Define a map $R : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ by $R(x, y) = (x + \alpha, y + \alpha)$ for an irrational α. Let μ be the Lebesgue measure on $\mathbb{T} \times \mathbb{T}$.

(a) Show that for any set of the form $C = A \times B$ with A, B Lebesgue measurable subsets of $\mathbb{T}$ (such a set is called a measurable rectangle) we have

$$R^{-1}C = C \implies \mu(C) = 0 \text{ or } \mu(C) = 1.$$

(b) Show that the transformation R is not ergodic with respect to μ.

Problem 5

(a) For $\alpha_1, \alpha_2 \in \mathbb{R}$ define $R_{\alpha_1}, R_{\alpha_2} : \mathbb{T} \to \mathbb{T}$ by $R_{\alpha_1}(x) = x + \alpha_1$ and $R_{\alpha_2}(x) = x + \alpha_2$. Find an arithmetic condition on α_1 and α_2 that is equivalent to the ergodicity of $R_{\alpha_1} \times R_{\alpha_2} : \mathbb{T} \times \mathbb{T} \to \mathbb{T} \times \mathbb{T}$ with respect to $m_{\mathbb{T}} \times m_{\mathbb{T}}$.

(b) Generalize part (a) to characterize ergodicity of the rotation $R_{\alpha_1} \times \cdots \times R_{\alpha_n} : \mathbb{T}^n \to \mathbb{T}^n$ for $(\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$.