Problem sheet 6

Problem 1

For a number $x \in (0,1)$, we write $x = 0.d_1(x)d_2(x)\dots$ for the decimal expansion of x.

(a) Show that for Lebesgue almost every x, the limit of

$$D(x) = \lim_{n\to\infty} \frac{d_1(x) + \cdots + d_n(x)}{n}$$

exists and compute it.

(b) Give examples of $x \in (0,1)$ such that $D(x)$ does not exist and such that $D(x) = \frac{1}{2024}$.

Problem 2

Recall that the *Gauss map* $G : [0,1] \to [0,1]$ is defined as follows:

$$G(x) := \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} - n & \text{if } x \in (\frac{1}{n+1}, \frac{1}{n}],\ n \geq 2. \end{cases}$$

(a) Show that $\frac{1}{\log 2}\frac{dx}{1+x}$ is a G-invariant probability measure on $[0,1]$.

(b) For a number $x \in (0,1)$, we write $x = [a_1(x), a_2(x), ...]$ for the continued fraction expansion of x, i.e.

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots}}}, \quad a_1(x), a_2(x), \ldots \in \mathbb{N}.$$

Show that for Lebesgue a.e. x,

$$\frac{a_1(x) + a_2(x) + \cdots + a_n(x)}{n} \to \infty \text{ as } n \to \infty,$$

but

$$\lim_{n\to\infty} \big(a_1(x)\cdots a_n(x)\big)^{\frac{1}{n}}$$

exists.

Problem 3

Let (X, A, μ) be a measure space with $\mu(X) = 1$, and $T : X \to X$ be a measure-preserving map. For $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, consider the linear map

$$U_\theta : L^2(X) \to L^2(X) : f \mapsto e^{i\theta}(f \circ T).$$

(a) Show that for every $f \in L^2(X)$,

$$\left\| \frac{1}{N} \sum_{k=0}^{N-1} U_\theta^k f - P_\theta f \right\|_2 \to 0,$$

where P_θ denotes the orthogonal projection to the subspace

$$V_\theta = \{f \in L^2(X) : f \circ T = e^{-i\theta} f \text{ a.e.}\}.$$

(Hint: Review the proof of the Mean Ergodic Theorem.)

(b) Suppose that T is mixing. Show that $V_\theta = 0$ and conclude that for every $f \in L^2(X)$,

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{ik\theta}(f \circ T^k) \to 0 \text{ in } L^2(X).$$

Problem 4

For $n \geq 2$ let A be an integral $n \times n$ matrix with determinant ± 1. Let

$$T : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n : x \mapsto Ax \pmod{\mathbb{Z}^n}$$

be the corresponding torus automorphism. Suppose that T is hyperbolic, i.e. A has no eigenvalue of modulus 1. Using Fourier series, show that

(a) T is ergodic with respect to the Lebesgue measure.

(b) T is mixing with respect to the Lebesgue measure.

Problem 5

(Kac's Lemma) Let $T : X \to X$ be an ergodic measure preserving transformation of a probability space $(X, \mathcal{B}, \mu)$ and let $A \subset X$ be a measurable subset with $\mu(A) > 0$. By Poincaré recurrence theorem, the integer

$$n_A(x) = \inf\{n \geq 1 : T^n(x) \in A\}$$

is defined for a.e. $x \in A$. For any $A \in \mathcal{B}$ be such that $\mu(A) > 0$, prove that $\int_A n_A d\mu = 1$.