Problem sheet 7

Problem 1

Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. We say that T is *totally ergodic* if T^n is ergodic for all $n \geq 1$. Given $K \geq 1$ define a space $X(K) = X \times \{1, \ldots, K\}$ with measure $\mu^{(K)} = \mu \times \nu$ defined on the product σ-algebra $\mathcal{B}^{(K)}$, where $\nu(A) = \frac{1}{K}|A|$ is the normalized counting measure defined on any subset $A \subseteq \{1, \ldots, K\}$, and a $\mu^{(K)}$-preserving transformation $T^{(K)}$ by

$$T^{(K)}(x, i) := \begin{cases} (x, i+1) & \text{if } i \in \{1, \ldots, K-1\}, \\ (Tx, 1) & \text{if } i = K. \end{cases}$$

for all $x \in X$. Show that $T^{(K)}$ is ergodic with respect to $\mu^{(K)}$ if and only if T is ergodic with respect to μ, and that $T^{(K)}$ is not totally ergodic if $K > 1$.

Problem 2

Let $X = [0, 1]$ and $T : X \to X$ be a continuous map.

(a) A pair of T-invariant probability measures μ, ν is called *singular* if there exists measurable $A \subset X$ such that

$$\mu(A) = 1 \quad \text{and} \quad \nu(A) = 0.$$

Show that if $\mu \neq \nu$ are ergodic, then they are singular. (Hint: Ergodic Theorem.)

(b) For T-invariant probability measures μ and ν, ν is called *absolutely continuous* with respect to μ if for every measurable $A \subset X$,

$$\mu(A) = 0 \quad \Longrightarrow \quad \nu(A) = 0.$$

Suppose that there exists $C > 0$ so that $\int_X f d\nu \leq C \int_X f d\mu$ for all $f \in C(X)$ with $f \geq 0$. Show that then ν is absolutely continuous with respect to μ. (Hint: Show first that the estimate is also true for $f = \chi_{[a,b]}$.)

(c) Suppose that there exists an ergodic, T-invariant measure μ on X, such that for every $x \in X$, there exists $C(x) > 0$ so that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(T^n x) \leq C(x) \int_X f \, d\mu$$

for all $f \in C(X)$ with $f \geq 0$.
Show that then μ is the unique ergodic measure, i.e. that there is no other ergodic, T-invariant measure.

Problem 3

Let $T : X \to X$ be an ergodic measure preserving transformation and $A \subset X$ a measurable subset of X of positive measure. Recall that the integer

$$n_A(x) = \inf\{n \geq 1 : T^n(x) \in A\}$$

is defined for a.e. $x \in A$. A is a probability space with measure $\mu_A := \mu/\mu(A)$. We define the induced or derivative transformation $T_A : A \to A$ by

$$T_A(x) = T^{n_A(x)}(x)$$

for a.e. $x \in A$. Prove that T_A is an ergodic measure preserving transformation with respect to μ_A.

Problem 4

Let $T\colon [0,1] \to [0,1],\ x \mapsto 4x(1-x)$. The goal of this exercise is to find a T-invariant measure. For a fixed L^1-function $\rho\colon [0,1] \to [0,\infty)$, we define the measure μ by

$$\mu(B) = \int_B \rho(x)\,dx,$$

for all Borel measurable subsets $B \subset [0,1]$.

(a) For $a \in [0,1] \to [0,1]$, denote by $x_1(a) < x_2(a)$ the roots of the polynomial $4x(1-x) = a$.
Show that if ρ satisfies

$$\int_0^a \rho(x)\,dx = \int_0^{x_1(a)} \rho(x)\,dx + \int_{x_2(a)}^1 \rho(x)\,dx$$

for all $a \in [0,1]$, then μ is T-invariant.

(b) Show that if ρ satisfies

$$\rho(a) = \frac{1}{4\sqrt{1-a}}\Big(\rho(x_1(a)) + \rho(x_2(a))\Big)$$

for all $a \in [0,1]$, then μ is T-invariant.

(c) Deduce that $\rho(x) = \frac{1}{\sqrt{x(1-x)}}$ gives a T-invariant measure μ.

Problem 5

Let $X = \mathbb{R}/\mathbb{Z}$ and $T\colon X \to X$ be defined by

$$T(x) = \begin{cases} 2x^2 + 1/2 & x \in [0, 1/2), \\ (2x-1)^2/2, & x \in [1/2, 1]. \end{cases}$$

(a) Show that for every $x \in (0,1) \setminus \{1/2\}$,

$$T^2(x) < x.$$

(b) Show that for every T-invariant measure μ on X,

$$\mu(X \setminus \{0, 1/2\}) = 0$$

and then deduce that T is uniquely ergodic.