Problem sheet 8

Problem 1

Let $T\colon \mathbb{R} \to \mathbb{R}$ be given by

$$T(x) = \frac{1}{2}\left(x - \frac{1}{x}\right)$$

for $x \neq 0$ and $T(0) = 0$.

(a) Show that the map T is two to one.

(b) Show that the map T is measure-preserving with respect to the measure μ defined by
$$\mu([a,b]) = \int_a^b \frac{1}{1+x^2} dx.$$

(c) Let ϕ be the map from $\mathbb{R}$ to $\mathbb{T}$ given by
$$\phi(x) = \frac{1}{\pi}\arctan(x) + \frac{1}{2}.$$
Show that ϕ is an isomorphism between the measure preserving systems $(\mathbb{R}, \mu, T)$ and $(\mathbb{T}, Leb, T_2)$ where T_2 is the doubling map $x \mapsto 2x($ mod 1) on the torus $\mathbb{T}$ and Leb is the Lebesgue measure.

Problem 2

Let $T_1, T_2\colon X \to X$ be measurable maps. Let μ_1 and μ_2 be T_1- and T_2-invariant probability measures on X, respectively. Define

$$T\colon X \times X \to X \times X,\ (x_1, x_2) \mapsto (T_1x_1, T_2x_2)$$

to be the product map and let μ be the product measure of μ_1 and μ_2.
Show that T is mixing for μ if and only if T_i is mixing for μ_i for $i = 1, 2$.

Assume now that $T_1 = T_2$ and $\mu_1 = \mu_2$. Is T also mixing with respect to the pushforward measure $\triangle_*\mu_1$ with respect to the diagonal embedding $\triangle\colon X \to X \times X,\ x \mapsto (x, x)$?

Problem 3

Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving system on a probability space and write $T^{(k)}$ for the k-fold Cartesian product $T \times \cdots \times T$.
Prove that $T^{(k)}$ is ergodic for all $k \geq 2$ if and only if $T^{(2)}$ is ergodic.
Does the same statement hold if we replace $T^{(2)}$ by $T^{(1)}$?

Problem 4

Let $(X, \mathscr{B}, \mu, T)$ be a measure preserving system on a probability space.

(a) Show that if this system has the property that for any $A, B \in \mathscr{B}$ there exists an N such that

$$\mu(A \cap T^{-n}B) = \mu(A)\mu(B)$$

for all $n \geq N$, then the system is trivial in the sense that $\mu(A) = 0$ or 1 for every $A \in \mathscr{B}$.

(b) Show that if this system has the property that the convergence

$$\mu(A \cap T^{-n}B) \to \mu(A)\mu(B)$$

is uniformly as $n \to \infty$ for any $A, B \in \mathscr{B}$, then the system is trivial as well.

Problem 5

Show that a $(X, \mathscr{B}, \mu, T)$ be a measure preserving system is weak-mixing if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} |\langle U_T^n f, g\rangle - \langle f, 1\rangle \cdot \langle 1, g\rangle| = 0$$

for all $f, g \in L^2_\mu(X)$, where U_T is the Koopman operator, i.e. $U_T f = f \circ T$ for all $f \in L^2_\mu(X)$.

Problem 6

Prove that a measure-preserving transformation T is weak-mixing if and only if for any measurable sets A, B and C with positive measure, there exists some $n \geq 1$ such that

$$T^{-n}A \cap B \neq \emptyset \quad \text{and} \quad T^{-n}A \cap C \neq \emptyset.$$