Problem sheet 9

Problem 1

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and consider the following map

$$T\colon \mathbb{T}^2 \to \mathbb{T}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 2x_2 + \alpha \\ x_2 + \alpha \end{pmatrix}.$$

Proof that the Lebesgue measure $m_{\mathbb{T}^2}$ is T-invariant and that T is ergodic with respect to $m_{\mathbb{T}^2}$.

Problem 2

Let $X = \{z \in \mathbb{C} \mid |z - i| = 1\}$. We define the stereographic projection π from X to the real axis by continuing the line from $2i$ through a unique point on $X \setminus \{2i\}$ until it meets the real axis.
The "North-South" map $T\colon X \to X$ is defined by

$$T(x) = \begin{cases} 2i & \text{if } z = 2i, \\ \pi^{-1}(\frac{\pi(x)}{2}) & \text{if } z \neq 2i. \end{cases}$$

Describe all T-invariant measures on X and find all ergodic ones.

Problem 3

This exercise illustrates that weak*-limits of ergodic measures on the full shift $\sigma\colon X = \{0,1\}^{\mathbb{Z}} \to X$ need not be ergodic.
Pick $x \in X$ such that any finite block of symbols of length l appeares in x with asymptotic frequency 2^{-l} – such points exist: indeed, the ergodic theorem says that almost every point satisfies this with respect to the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure. For $n \in \mathbb{N}$, define $y^{(n)} \in X$ determined by the two conditions

- $y^{(n)}|_{[0,2n-1]} = x_1 x_2 \dots x_n 0 \dots 0$,
- $\sigma^{2n}(y^{(n)}) = y^{(n)}$.

For each $n \in \mathbb{N}$ construct an ergodic σ-invariant measure μ_n on X, supported on the orbit of the periodic point $y^{(n)}$.
Show that μ_n converges to some limit ν in the weak*-topology and use the characterization of ergodic measures seen in class, to deduce that ν is *not* ergodic.

Problem 4

Let $T\colon X \to X$ be a dynamical system which is uniquely ergodic, i.e. $\mathcal{M}(X)^T = \{\mu\}$ consists of a single element and μ is T-ergodic.
Proof that for any $f \in \mathcal{C}(X)$ there exists a constant C_f so that

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) \xrightarrow[n\to\infty]{} C_f$$

uniformly in $x \in X$.
Hint: Proof by contradiction - construct a certain T-invariant ergodic measure different from μ.

Problem 5

Consider the doubling map $T_2\colon \mathbb{T} \to \mathbb{T},\ x \mapsto 2x \mod 1$ and equip $\mathbb{T}$ with the Lebesgue measure $m_{\mathbb{T}}$.

(a) Construct a point that is generic for $m_{\mathbb{T}}$.

(b) Construct a point that is generic for an ergodic T_2-invariant ergodic measure other than $m_{\mathbb{T}}$.

(c) Construct a point that is generic for a non-ergodic T_2-invariant measure.

(d) Construct a point that is *not* generic for any T_2-invariant measure.

Problem 6

A homeomorphism $T\colon X \to X$ of a compact metric space is called *topologically ergodic* if every closed proper T-invariant subset of X has empty interior. Show that the following properties are equivalent:

(a) (X, T) is topologically ergodic,

(b) there is a point in X with a dense orbit,

(c) for any non-empty open sets O_1 and O_2 in X, there is some $n \geq 0$ for which $O_1 \cap T^n O_2 \neq \emptyset$.