ANALYSIS IV MOCK EXAM

Exercise 0.1. (1) Give the definition of inner product space and Hilbert space over $\mathbb{C}$.
(2) What is an orthonormal system of a Hilbert space? State and prove Bessel inequality.
(3) For a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and a closed subspace $Y \subset H$, define the nearest point projection of a point x onto Y and show that the projection is unique.
(4) Let $H = L^2(\mathbb{R}, \mathbb{C})$ and let V be the subspace generated by the function $g(x) = xe^{-x^2}$. Give an explicit formula for the projection onto V.[1]

Exercise 0.2. (1) Give the definition of Fourier transform in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and show that if $\varphi \in \mathcal{S}(\mathbb{R}^d)$, also $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$
(2) Show that for any function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ there exists another function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\hat{\psi} = \varphi$. Is it also unique?
(3) Compute the Fourier transform of $f(x) = \chi_{[1,3]}(x)$ (characteristic function of the interval $[1, 3]$).

Exercise 0.3. Consider the heat-type PDE

$$\begin{cases} u_t = e^{-t}u + u_{xx} & (t, x) \in (0, +\infty) \times \mathbb{R}, \\ u(0, x) = f(x) & \text{in } \mathbb{R}, \end{cases} \tag{P}$$

where u is assumed to be a real-valued 2π-periodic function on $\mathbb{R}$ and f is also 2π-periodic.

- Assuming that you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ of f, construct a formal solution w to (P) as a Fourier series in the x variable with t dependent coefficients.
- Check that if $f \in L^2([-\pi, \pi])$ the function w constructed is well-defined, of class C^2, real-valued and solves

$$w_t = e^{-t}w + w_{xx} \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}$$

- Show that the initial condition is met, in the sense that

$$\lim_{t\to 0^+} \|w(t, \cdot) - f\|_{L^2(-\pi,\pi)} = 0.$$

[1] No need to find explicit values for coefficients, integral expressions are sufficient.

REMARK: You can use the results seen in class if you clearly identify them, by either using their name or stating unambiguously the assumptions and the conclusion. You can also give for granted the following definitions and use the facts below without reproving them.

- Vector space over $\mathbb{C}$.
- Completeness in normed spaces.
- Schwartz space in $\mathbb{R}^d$.
- The formula for Fourier transform of a derivative and derivative of the Fourier transform.
- The Fourier inversion formula for Schwartz functions.
- The formula $\mathcal{F}(\chi_{[-1,1]}) = \sqrt{2/\pi}(\sin x)/x$.
- Parseval's identity for Fourier series.