ANALYSIS IV MOCK EXAM

Exercise 0.1. (1) Give the definition of inner product space and Hilbert space over $\mathbb{C}$.

(2) What is an orthonormal system of a Hilbert space? State and prove Bessel inequality.

(3) For a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and a closed subspace $Y \subset H$, define the nearest point projection of a point x onto Y and show that the projection is unique.

(4) Let $H = L^2(\mathbb{R}, \mathbb{C})$ and let V be the subspace generated by the function $g(x) = xe^{-x^2}$. Give an explicit formula for the projection onto V.[1]

Solution:

(1) An inner product space is a pair $(V, \langle\cdot,\cdot\rangle)$, where V is a vector space over $\mathbb{C}$ and $\langle\cdot,\cdot\rangle$ satisfies
- conjugate symmetry: $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for every $v, w \in V$;
- linearity in the first argument: $\langle \alpha v + \beta w, z\rangle = \langle \alpha v, z\rangle + \langle \beta w, z\rangle$ for every $\alpha, \beta \in \mathbb{C}$ and $v, w, z \in V$;
- positive definiteness: for every $v \neq 0$, $\langle v, v\rangle > 0$.

A Hilbert space is an inner product space that is complete with respect to the norm induced by the inner product $\|v\| = \sqrt{\langle v, v\rangle}$.

(2) An orthonormal system is a set $\{e_i\}_{i\in\mathbb{N}}$ such that $\langle e_i, e_j\rangle = \delta_{ij}$ for any $i, j \in \mathbb{N}$. Bessel inequality states that

$$\sum_{i\in\mathbb{N}} |\langle e_i, v\rangle|^2 \leq \|v\|^2.$$

Call $S_n(v) = \sum_{i=1}^n \langle e_i, v\rangle e_i$ and compute

$$0 \leq \|S_n(v) - v\|^2 = \|S_n(v)\|^2 - 2\Re(\langle S_n(v), v\rangle) + \|v\|^2.$$

Observe that

$$\|S_n(v)\|^2 = \sum_{i=1}^n |\langle v, e_i\rangle|^2 = \sum_{i=1}^n \langle v, e_i\rangle\langle e_i, v\rangle = \langle S_n(v), v\rangle$$

thus we find

$$0 \leq \|v\|^2 - \|S_n(v)\|^2.$$

[1] No need to find explicit values for coefficients, integral expressions are sufficient.

(3) For any $x \in H$ the nearest point projection $\pi_Y(x)$ of x onto Y is an element of Y satisfying

$$\|x - \pi_Y(x)\| = \min_{y \in Y} \|x - y\|.$$

Call the minimum value d. If there were two elements $y, y' \in Y$ solving the minimisation problem, then by the parallelogram law

$$\begin{aligned}\|y - y'\| &= 2\|x - y\| + 2\|x - y'\| - 4\|x - (y + y')/2\| \\ &= 4d - 4\|x - (y + y')/2\|\end{aligned}$$

Since $(y + y')/2 \in Y$, we have $\|x - (y + y')/2\| \geq d$ and thus

$$\|y - y'\| \leq 0,$$

which implies $y = y'$.

(4) The projection onto a subspace V generated by a single element g is given by

$$\pi_V(f) = \frac{\langle f, g\rangle}{\|g\|^2} g$$

So, in our case,

$$\pi_V(f)(y) = \frac{\int_{\mathbb{R}} f(x) x e^{-x^2} dx}{\int_{\mathbb{R}} x^2 e^{-2x^2} dx} y e^{-y^2}$$

Exercise 0.2. (1) Give the definition of Fourier transform in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and show that if $\varphi \in \mathcal{S}(\mathbb{R}^d)$, also $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$

(2) Show that for any function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ there exists another function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\hat{\psi} = \varphi$. Is it also unique?

(3) Compute the Fourier transform of $f(x) = \chi_{[1,3]}(x)$ (characteristic function of the interval $[1, 3]$).

Solution:

(1) For any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ set

$$\hat{\varphi}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(x) e^{-i\xi \cdot x} dx$$

We need to show that for any multi-indices α, β we have

$$\xi^\alpha D_\xi^\beta \hat{\varphi}(\xi) \in L^\infty(\mathbb{R}^d).$$

We have

$$\left|\xi^\alpha D_\xi^\beta \hat{\varphi}(\xi)\right| = \left|\mathcal{F}\left(D^\alpha (x^\beta \varphi)\right)(\xi)\right|$$

and thus

$$\left\|\xi^\alpha D_\xi^\beta \hat{\varphi}(\xi)\right\|_\infty = \left\|\mathcal{F}\left(D^\alpha (x^\beta \varphi)\right)(\xi)\right\|_\infty \leq C \|D^\alpha (x^\beta \varphi)\|_{L^1}$$

and the last term is finite for any α, β since $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

(2) Using the inversion formula it is enough to define

$$\psi(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \varphi(x) e^{i\xi \cdot x} dx.$$

To show that $\psi \in \mathcal{S}(\mathbb{R}^d)$, we can set $\varphi'(x) = \varphi(-x) \in \mathcal{S}(\mathbb{R}^d)$ and observe that $\psi = \hat{\varphi}'$ and use point (1). Uniqueness follows from Plancherel's identity once we observe that if we had two functions ψ, ψ' with the same transform, then $\mathcal{F}(\psi - \psi') = 0$.

(3) Denoting $\tau_h f := f(\cdot - h)$ recall that that $\mathcal{F}(\tau_h f) = e^{-ih\cdot\xi} \hat{f}(\xi)$. In our case we have $f(x) = \tau_2 \chi_{[-1,1]}$, thus

$$\hat{f} = e^{-i2\xi} \mathcal{F}(\chi_{[-1,1]}) = \sqrt{2/\pi} e^{-i2\xi} (\sin \xi)/\xi$$

Exercise 0.3. Consider the heat-type PDE

$$\begin{cases} u_t = e^{-t} u + u_{xx} & (t, x) \in (0, +\infty) \times \mathbb{R}, \\ u(0, x) = f(x) & \text{in } \mathbb{R}, \end{cases} \tag{P}$$

where u is assumed to be a real-valued 2π-periodic function on $\mathbb{R}$ and f is also 2π-periodic.

- Assuming that you are given the Fourier coefficients $\{c_k(f)\}_{k\in\mathbb{Z}}$ of f, construct a formal solution w to (P) as a Fourier series in the x variable with t dependent coefficients.
- Check that if $f \in L^2([-\pi, \pi])$ the function w constructed is well-defined, of class C^2, real-valued and solves

$$w_t = e^{-t} w + w_{xx} \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}$$

- Show that the initial condition is met, in the sense that

$$\lim_{t \to 0^+} \|w(t, \cdot) - f\|_{L^2(-\pi,\pi)} = 0.$$

Solution: we write formally

$$w(t, x) = \sum_{k\in\mathbb{Z}} w_k(t) e^{-ikx}.$$

Plugging in w in (P), after formally differentiating, we get to an equation for w_k of the form

$$w_k' = \left(e^{-t} - k^2\right) w_k$$

whereas the boundary conditions follow from the formal equality

$$\sum_{k\in\mathbb{Z}} c_k(f) e^{-ikx} = f(x) = w(0, x) = \sum_{k\in\mathbb{Z}} w_k(0) e^{-ikx}.$$

Integrating we find
$$w_k(t) = c_k(f)\exp(-(e^{-t} + k^2 t))$$
and the formal solution is given by
$$w(t,x) \coloneqq \sum_{k\in\mathbb{Z}} c_k(f)\exp(-(e^{-t} + k^2 t))e^{-ikx}.$$
Next, we check that for any $\delta > 0$ and any integers $p, q \geq 0$ with $p + q \leq 2$ we have
$$\sum_{k\in\mathbb{Z}} \sup_{(\delta,1/\delta)\times\mathbb{R}} |\partial_t^p \partial_x^q (w_k(t)e^{-ikx})| < \infty$$
This will prove that the series converges uniformly, along with its derivatives up to order 2, on compact subsets of $(0, +\infty)\times\mathbb{R}$ and therefore that it is a classical solution since differentiation can be performed term by term. We have, for $(t, x) \in (\delta, 1/\delta)\times\mathbb{R}$,
$$|\partial_t^p \partial_x^q \exp(-(e^{-t} + k^2 t + ikx))| \leq C k^{2p+q} e^{-k^2\delta}$$
so that, using Cauchy–Schwarz and Parseval's identity,
$$\begin{aligned}\sum_{k\in\mathbb{Z}} \sup_{(\delta,1/\delta)\times\mathbb{R}} |\partial_t^p \partial_x^q (w_k(t)e^{-ikx})| &\leq C \left(\sum_{k\in\mathbb{Z}} |c_k(f)|^2\right)^{1/2} \left(\sum_{k\in\mathbb{Z}} k^{4p+2q} e^{-2k^2\delta}\right)^{1/2} \\ &\leq C' \|f\|_{L^2} \left(\sum_{k\in\mathbb{Z}} k^{4p+2q} e^{-2k^2\delta}\right)^{1/2}\end{aligned}$$
which is summable. Finally, applying Parseval's identity again we find
$$\begin{aligned}\lim_{t\to 0^+} \|w(t,\cdot) - f\|^2_{L^2(-\pi,\pi)} &= \lim_{t\to 0^+} \sum_{k\in\mathbb{Z}} |c_k(f)\exp(-(e^{-t} + k^2 t)) - c_k(f)|^2 \\ &= \lim_{t\to 0^+} \sum_{k\in\mathbb{Z}} |c_k(f)|^2 |\exp(-(e^{-t} + k^2 t)) - 1|^2 = 0\end{aligned}$$
The last step holds by dominated convergence theorem, after observing that the last series is an integral in $L^1(\mathbb{Z}, P(\mathbb{Z}), \sharp)$ of
$$\phi_t(k) \coloneqq |c_k(f)|^2 |\exp(-(e^{-t} + k^2 t)) - 1|^2$$
which are bounded, uniformly in t, by $4|c_k(f)|^2 \in L^1(\mathbb{Z}, P(\mathbb{Z}), \sharp)$.

REMARK: You can use the results seen in class if you clearly identify them, by either using their name or stating unambiguously the assumptions and the conclusion. You can also give for granted the following definitions and use the facts below without reproving them.

- Vector space over $\mathbb{C}$.
- Completeness in normed spaces.
- Schwartz space in $\mathbb{R}^d$.
- The formula for Fourier transform of a derivative and derivative of the Fourier transform.
- The Fourier inversion formula for Schwartz functions.
- The formula $\mathcal{F}(\chi_{[-1,1]}) = \sqrt{2/\pi}(\sin x)/x$.
- Parseval's identity for Fourier series.