These closed-answer questions cover some topics from previous classes that will be useful for this course. If you find some of them obscure you are encouraged to revise briefly the relative topic and/or to come to office hours.

0.1. Inclusion between L^p spaces.

1. Is it true that if $f \in L^1(\mathbb{R}^n)$ then necessarily $f \in L^2(\mathbb{R}^n)$?
2. Is it true that if $f \in \ell^1(\mathbb{N})$ then necessarily $f \in \ell^2(\mathbb{N})$?
3. Give an example of a function that is $L^{99}(0,1)$ but not $L^{100}(0,1)$. Could you find one also in $L^{100}(0,1) \setminus L^{99}(0,1)$?

0.2. Exchanging limits and integrals.

1. Recall that the Dominated Convergence Theorem implies that a collection of measurable functions $f_n \colon \mathbb{R} \to \mathbb{C}$, satisfying $|f_n| \leq g$ for some $g \in L^1(\mathbb{R})$, also satisfies

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n(x)\right) dx$$

   whenever the pointwise limit $\lim_{n\to\infty} f_n(x)$ exists a.e. Show, via a counterexample, that the hypothesis $|f_n| \leq g \in L^1$ is necessary. **Hint:** can you think of an example in which the statement fails? For instance, a sequence of functions f_n with constant integral (> 0) but with poitwise limit 0?

2. Let $(f_n)_{n\in\mathbb{N}}$ be a collection of non-negative measurable functions. Is it true that

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x)dx?$$

0.3. Completeness and Cauchy sequences.

1. Is it true that a Cauchy sequence (say, in a metric space) can have at most one limit?
2. Is it true that the interval $(0,1) \subset \mathbb{R}$ is complete?
3. Consider the sequence of functions $f_n \colon (0,+\infty) \to \mathbb{R}$ defined by

$$f_n(x) = \frac{\tanh(x)}{x}\chi_{(0,n)}(x).$$

   Determine the pointwise limit f and discuss the convergence $f_n \to f$ in L^2. Is the limit also in L^1? What can we deduce about the completeness of L^1 with respect to $\|\cdot\|_{L^2}$?

4. Can you build a sequence of functions $\{f_k\} \subset L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |f_k(x) - 1|^2 \, dx \to 0 \text{ as } k \to \infty\,?$$

Hint: Use the triangle inequality on $\|\cdot\|_{L^2}$ to show that is a sequence of functions $f_n \in L^2$ converges in L^2 to f, then $f \in L^2$.

0.4. Approximability in normed spaces. Denote with $C_c(X)$ the space of continuous functions with compact support in X, that is

$$\{f : \overline{\{f \neq 0\}} \text{ is compact in } X\}$$

and denote with $S(X)$ the class of step functions defined on X. Determine whether the following statements are true or false and justify your answer. The statement $X \subseteq \overline{Y}^Z$ should be interpreted as "all elements in X can be approximated by elements of Y in the topology of Z".

1. $L^1(0,1) \subseteq \overline{C([0,1])}^{L^1}$
2. $L^\infty(0,1) \subseteq \overline{C([0,1])}^{L^\infty}$
3. $L^1(0,1) \subseteq \overline{S([0,1])}^{L^1}$
4. $L^\infty(0,1) \subseteq \overline{S([0,1])}^{L^\infty}$
5. $L^\infty(0,1) \subseteq \overline{C([0,1])}^{L^1}$
6. $C((0,1)) \subseteq \overline{C_c((0,1))}^{L^\infty}$