These closed-answer questions cover some topics from previous classes that will be useful for this course. If you find some of them obscure you are encouraged to revise briefly the relative topic and/or to come to office hours.

0.1. Inclusion between L^p spaces.

1. Is it true that if $f \in L^1(\mathbb{R}^n)$ then necessarily $f \in L^2(\mathbb{R}^n)$?
2. Is it true that if $f \in \ell^1(\mathbb{N})$ then necessarily $f \in \ell^2(\mathbb{N})$?
3. Give an example of a function that is $L^{99}(0,1)$ but not $L^{100}(0,1)$. Could you find one also in $L^{100}(0,1) \setminus L^{99}(0,1)$?

0.2. Exchanging limits and integrals.

1. Recall that the Dominated Convergence Theorem implies that a collection of measurable functions $f_n \colon \mathbb{R} \to \mathbb{C}$, satisfying $|f_n| \leq g$ for some $g \in L^1(\mathbb{R})$, also satisfies

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n(x) \right) dx$$

   whenever the pointwise limit $\lim_{n\to\infty} f_n(x)$ exists a.e. Show, via a counterexample, that the hypothesis $|f_n| \leq g \in L^1$ is necessary. **Hint:** can you think of an example in which the statement fails? For instance, a sequence of functions f_n with constant integral (> 0) but with poitwise limit 0?

2. Let $(f_n)_{n\in\mathbb{N}}$ be a collection of non-negative measurable functions. Is it true that

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} f_n(x)dx?$$

0.3. Completeness and Cauchy sequences.

1. Is it true that a Cauchy sequence (say, in a metric space) can have at most one limit?
2. Is it true that the interval $(0,1) \subset \mathbb{R}$ is complete?
3. Consider the sequence of functions $f_n \colon (0,+\infty) \to \mathbb{R}$ defined by

$$f_n(x) = \frac{\tanh(x)}{x} \chi_{(0,n)}(x).$$

   Determine the pointwise limit f and discuss the convergence $f_n \to f$ in L^2. Is the limit also in L^1? What can we deduce about the completeness of L^1 with respect to $\| \cdot \|_{L^2}$?

4. Can you build a sequence of functions $\{f_k\} \subset L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |f_k(x) - 1|^2 \, dx \to 0 \text{ as } k \to \infty\,?$$

   Hint: Use the triangle inequality on $\|\cdot\|_{L^2}$ to show that is a sequence of functions $f_n \in L^2$ converges in L^2 to f, then $f \in L^2$.

0.4. Approximability in normed spaces. Denote with $C_c(X)$ the space of continuous functions with compact support in X, that is

$$\{f : \overline{\{f \neq 0\}} \text{ is compact in } X\}$$

and denote with $S(X)$ the class of step functions defined on X. Determine whether the following statements are true or false and justify your answer. The statement $X \subseteq \overline{Y}^Z$ should be interpreted as "all elements in X can be approximated by elements of Y in the topology of Z".

1. $L^1(0,1) \subseteq \overline{C([0,1])}^{L^1}$
2. $L^\infty(0,1) \subseteq \overline{C([0,1])}^{L^\infty}$
3. $L^1(0,1) \subseteq \overline{S([0,1])}^{L^1}$
4. $L^\infty(0,1) \subseteq \overline{S([0,1])}^{L^\infty}$
5. $L^\infty(0,1) \subseteq \overline{C([0,1])}^{L^1}$
6. $C((0,1)) \subseteq \overline{C_c((0,1))}^{L^\infty}$

0. Solutions

Solution of 0.1:

1. No, there is no inclusion between these spaces. For example take $f(t) = |t|/(1+t^2)$ and $g(t) = e^{-t^2}|t|^{-3/4}$. Then $f \in L^2 \setminus L^1$ and $g \in L^1 \setminus L^2$.
2. Yes it is true. Let $f \in \ell^1(\mathbb{N})$. Since an absolutely convergent sequence is infinitesimal we have that $f(j)^2 \leq |f(j)|$ for all $j \geq N$, for N large enough. Hence $\sum_j f(j)^2 \leq \sum_{j \leq N} f(j)^2 + \sum_{j \geq N} |f(j)| < \infty$.
3. Try using powers: let $f(x) = x^\alpha$. We require

$$f \in L^{99} \iff |f|^{99} \in L^1 \iff x^{99\alpha} \in L^1 \iff 99\alpha > -1,$$
$$f \notin L^{100} \iff |f|^{100} \notin L^1 \iff x^{100\alpha} \notin L^1 \iff 100\alpha \leq -1.$$

   Thus any $\alpha \in (-1/99, -1/100]$ works. The reverse question admits no solution by inclusion of L^p spaces on a bounded domain: $L^p(0,1) \subset L^q(0,1) \iff q < p$.

Solution of 0.2:

1. The functions $f_n(x) := n\chi_{(-1/n,1/n)}(x)$ provide a counterexample. Indeed, the pointwise limit is 0 a.e. and

$$\lim_n \int_{\mathbb{R}} f_n = 2 \neq 0 = \int_R f$$

   Thus, any function g dominating the sequence cannot be integrable. Indeed, one can always use the lowest dominant

$$h(x) := \sup_{n \in \mathbb{N}} f_n(x),$$

   which is bounded by any dominant g, to show that

$$\int_{\mathbb{R}} g \geq \int_{\mathbb{R}} h = 2\sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} n dx = 2\sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

2. Yes, as a direct consequence of the Monotone Convergence Theorem applied to the partial sums $g_N(x) = \sum_{n=1}^N f_n(x)$.

Solution of 0.3:

1. Yes, it is true. If (x_n) had two distinct limit points a, b they would have positive distance δ. But convergence to a implies that for n sufficiently large $d(x_n, a) < \delta/2$. But then

$$d(x_n, b) \geq d(a,b) - d(x_n, a) > \delta - \delta/2 = \delta/2,$$

   hence (x_n) cannot converge to b.

2. No, the sequence $x_j := 2^{-j}$ is Cauchy, but its limit point in $\mathbb{R}$ (i.e., zero) does not lie in $(0,1)$.
3. The pointwise limit is clearly $f(x) = \tanh(x)/x$ and the convergence in L^2 holds. Indeed
$$\int_0^{+\infty} |f_n - f|^2 = \int_n^{+\infty} \left|\frac{\tanh(x)}{x}\right|^2 dx \to 0 \quad \text{as } n \to +\infty$$
since, near $+\infty$ we have $\tanh(x)/x \sim 1/x \in L^2(+\infty)$, which implies that the integral of the tail (that is, the integral in $(n,+\infty)$) is infinitesimal with n large. Now, observe that $f_n \in L^1$ for every n and that the sequence is Cauchy in L^2, but the limit is not L^1 since near $+\infty$ we have $\tanh(x)/x \sim 1/x \notin L^1(+\infty)$. This proves that L^1 is not complete with respect to the L^2 distance.
4. Suppose we could construct such sequence; then, by the triangle inequality we have
$$\|f\|_{L^2} \leq \|f_k - f\|_{L^2} + \|f_k\|_{L^2} \leq 1 + \|f_k\|_{L^2}$$
if n is chosen sufficiently large. Since $f_k \in L^2$, this implies that $f \in L^2$, but $1 \notin L^2(\mathbb{R})$ and we get a contradiction.

Solution of 0.4:

1. Yes, it is a Theorem from Mass und Integral (see Theorem 3.7.15 in Da Lio's notes)
2. No, because uniform (L^∞) limit of continuous functions is continuous, but a generic L^∞ function is not continuous.
3. Yes, since $C([0,1]) \subset \overline{S([0,1])}^{L^\infty}$, i.e. step functions approximate continuous functions in the L^∞ topology. But in $[0,1]$ we have $\|\cdot\|_{L^1} \leq \|\cdot\|_{L^\infty}$, so step functions approximate continuous functions also in the L^1 topology. The conclusion follows from point 1.
4. No, since the function
$$f = \sum_{i=1}^{\infty} \chi_{\left(\frac{1}{2i}, \frac{1}{2i-1}\right)} \in L^\infty(0,1).$$
cannot be approximated in L^∞ by step functions. Indeed, since step functions are made by finite linear combination of characteristic functions of intervals, the L^∞ distance between a step function and f will always be ≥ 1.
5. This follows from 1, after observing that $L^\infty(0,1) \subset L^1(0,1)$.
6. No, for example the L^∞ distance between 1 and any $f \in C_c((0,1))$ is always ≥ 1.