The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

1.1. Inner product spaces.

1. Let $V \coloneqq \mathrm{M}_{n\times n}(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries and define the *Fobenius product* $\langle\cdot,\cdot\rangle\colon V\times V \to \mathbb{C}$ as

$$\langle A, B\rangle \coloneqq \mathrm{Tr}(AB^\dagger) = \sum_{i,j=1}^{n} a_{ij}\bar{b}_{ij}$$

where Tr denotes the trace and $B^\dagger$ is the Hermitian transpose of B, obtained by transposition and complex conjugation of the entries: $B^\dagger = \overline{B^T}$. Show that $(V, \langle\cdot,\cdot\rangle)$ is an inner-product space. **Hint:** first observe that $\mathrm{Tr}(A) = \overline{\mathrm{Tr}(A^\dagger)}$.

2. Consider n inner-product spaces $(V_1, \langle\cdot,\cdot\rangle_1), \dots, (V_n, \langle\cdot,\cdot\rangle_n)$. Is $(V, \langle\cdot,\cdot\rangle)$, where $V = V_1 \times \cdots \times V_n$ and

$$\langle (v_1,\dots,v_n),(w_1,\dots,w_n)\rangle \coloneqq \sum_{i=1}^{n}\langle v_i, w_i\rangle_i,$$

an inner product space?

3. Let $W \coloneqq M_{n\times n}(L^2(\mathbb{R},\mathbb{C}))$ be the space of $n\times n$ matrices whose entries are square integrable functions from $\mathbb{R}$ to $\mathbb{C}$. Which product would make W an inner product space? **Hint:** observe that W is a "composition" of two inner product spaces.

1.2. Continuity of operations. An inner product space $(V, \langle\cdot,\cdot\rangle)$ is also a metric space under the norm $|\cdot| := \sqrt{\langle\cdot,\cdot\rangle}$, hence it has a natural topology. Prove that $\langle\cdot,\cdot\rangle$ and the vector space operations $(\cdot, +)$ are continuous from $V\times V$ (resp. $V\times\mathbb{C}$, $V\times V$) endowed with the natural product topology, to $\mathbb{C}$ (resp. V, V). Recall that a natural topology in $V \times V$ is the one induced by $|\cdot|$, i.e. the one induced by the norm

$$|(v_1, v_2)|_{V\times V} := |v_1| + |v_2|.$$

Similarly, the norm (thus the metric and the topology) on $V\times\mathbb{C}$ is given by

$$|(v,\alpha)|_{V\times\mathbb{C}} := |v| + |\alpha|.$$

Hint: is there a clever way to write $\langle\cdot,\cdot\rangle$, in order to prove continuity?

1.3. Topology of normed spaces. Determine whether the following sets X are well-defined, open, close, subspaces and convex.

1. In the normed space $(C([0,1]), \|\cdot\|_{L^\infty})$, the subset X of nowhere vanishing functions.
2. In the normed space $(C([0,1]), \|\cdot\|_{L^2})$, the subset X of nowhere vanishing functions.
3. (BONUS) In the normed space $(L^2(0,1), \|\cdot\|_{L^2})$, the subset $X = \{f : \int_0^1 f = 1\}$.

   ☐ Not well defined.

   ☐ Well defined, open and convex.

   ☐ Well defined, closed, convex but not a linear subspace.

   ☐ Well defined, closed and linear subspace.
4. In the normed space $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$, the subset $\{f : f(x) = f(-x) \text{ for a.e. } x \in \mathbb{R}\}$.
   Hint: It's useful to recall that if $u_k \to u$ in L^2 then, up to picking a subsequence, there is a null measure set N such that $u_k(x) \to u(x)$ for all $x \notin N$.
5. $(*)$ In the normed space $(L^2(0,1), \|\cdot\|_{L^2})$, the subset $X = \{f : f \geq 0 \text{ and } \int_0^1 \frac{2f}{1+f} \geq 1\}$. **Hint:** observe that the map $s \mapsto 2s/(1+s)$ is concave for $s \geq 0$.

1.4. Quantitative Cauchy Schwarz. Let H be a real inner product space, prove the identity

$$|x||y| - x \cdot y = \frac{|x||y|}{2}\left|\frac{x}{|x|} - \frac{y}{|y|}\right|^2 \geq 0 \text{ for all } x, y, \in H.$$

Characterize the set $C \subset H \times H$ of pair of vectors that saturate the Cauchy-Schwarz inequality, i.e. $x \cdot y = |x||y|$. Plot C in the case $H = \mathbb{R}$.

$(*)$ If x, y are ϵ-close to saturate the Cauchy Schwarz inequality, that is

$$(1-\epsilon)|x||y| \leq x \cdot y,$$

then how close are x, y to the set C? Bound from above the number

$$\inf_{(x',y') \in C} |x - x'|^2 + |y - y'|^2 =: \operatorname{dist}^2((x,y), C).$$