The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

10.1. Some Fourier transforms. Compute the following one dimensional Fourier transforms for

$$e^{ix-|x|^2}, \quad e^{-a|x|}, \quad e^{-ax}\sin(3x)\mathbf{1}_{(0,\infty)},$$

where $a \in \mathbb{C}, \mathrm{Re}(a) > 0$.

(BONUS) Given $u, v \in \mathcal{S}(\mathbb{R})$ compute the Fourier transforms of

$$(x,y) \mapsto u(2x)v(y/2)$$

in terms of $\hat{u}, \hat{v}$.

10.2. Dominated convergence review. Motivate each of the following statements using the dominated convergence theorem in a suitable measure space, but pay attention: one of them is in fact *false*!

1. Given $f \in L^2(\mathbb{R}^d)$ it holds

$$\int_{\{|x|>R\}} f(x)^2 \sin(x_1)\, dx \to 0 \text{ as } R \to \infty$$

2. Given $f \in L^1(\mathbb{R}^d)$ it holds

$$\int_{\{|x|>R\}} \frac{\sqrt{1+f(x)^2}-1}{1+\hat{f}(x)}\, dx \to 0 \text{ as } R \to \infty$$

3. Let $\psi \in C^1_c(\mathbb{R}^d)$ such that $\psi(x) \equiv 1$ in a neighbourhood of $x = 0$. Then for each $f \in L^1(\mathbb{R}^d)$ it holds

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} f(x)\psi(\epsilon x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx \quad \text{and} \quad \lim_{\epsilon \to 0}\int_{\mathbb{R}^d} f(x)\partial_{x_j}\psi(\epsilon x)\, dx = 0.$$

4. Let $\{c_k\} \in \ell^1(\mathbb{N})$. The map $f(t) := \sum_{k\in\mathbb{N}} c_k e^{i\sin(k)t}$ is of class $C^\infty(\mathbb{R})$ and its derivatives are given by $f^{(m)}(t) = \sum_{k\in\mathbb{N}}(i\sin(k))^m c_k e^{i\sin(k)t}$.

5. Let $\{c_k\} \in \ell^2(\mathbb{N})$. The map $f(t) := \sum_{k\in\mathbb{N}} c_k e^{ikt^2}$ is of class $C^1(\mathbb{R})$ and its derivative is given by $f'(t) = 2it\sum_{k\in\mathbb{N}} kc_k e^{ikt^2}$.

10.3. Harmonic functions on the disk. In this problem we show the existence of the so-called harmonic extension in the interior of the disk of a sufficiently regular function f defined on the disk boundary.

Consider the second order differential operators in two variables (x_1, x_2):

$$\Delta := \partial_{11} + \partial_{22} \qquad \text{and} \qquad L := \partial_{11} + \frac{1}{x_1}\partial_1 + \frac{1}{x_1^2}\partial_{22}.$$

We say that a twice differentiable function $w(x_1, x_2)$ is harmonic $\Delta w = 0$ in its domain.

1. Given $u \colon \overline{D} \to \mathbb{R}$, where $D := \{(x, y) : x^2 + y^2 < 1\}$, consider the function
$$v(r,\theta) := u(r\cos\theta, r\sin\theta), \quad r \in [0,1], \theta \in \mathbb{R}.^1 \tag{1}$$
Using the chain rule, check that $(\Delta u)(r\cos\theta, r\sin\theta) = Lv(r,\theta)$ for all $r \in (0,1)$ and $\theta \in \mathbb{R}$.

2. Given any regular function $F \colon \partial D \to \mathbb{R}$ consider its 2π periodic version $f \colon \mathbb{R} \to \mathbb{R}$ defined by
$$F(\cos\theta, \sin\theta) =: f(\theta), \quad \theta \in \mathbb{R}.$$
Show that we can find a solution of
$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \{0\}, \\ u = F & \text{on } \partial D, \end{cases}$$
solving instead
$$\begin{cases} \partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v = 0 & \text{in } (0,1] \times \mathbb{R}, \\ v(r, \theta + 2\pi) = v(r,\theta) & \text{in } (0,1] \times \mathbb{R}, \\ v(1,\theta) = f(\theta) & \text{for all } \theta \in \mathbb{R}, \end{cases} \tag{2}$$
and then defining u trough (1).

3. Formally solve the system (2) by the Ansatz $v := \sum_{k\in\mathbb{Z}} u_k(r) e^{ik\theta}$. Explain why the $\{u_k(r)\}$ are not *uniquely* determined by the $\{c_k(f)\}$. Explain why they are unique if we further require that
$$\limsup_{r\downarrow 0} |u_k(r)| < \infty \quad \forall k \in \mathbb{Z}. \tag{3}$$

4. Let $v(r,\theta)$ be the Ansatz constructed in the previous point using the extra assumption (3). Show that v is of class C^∞ in the (r,θ) variables in $[0,1) \times \mathbb{R}$, as soon as $f \in L^2$.

5. $(*)$ show that, as soon as $f \in L^2(-\pi,\pi)$, the v you constructed with the extra assumption (3), corresponds in fact to a u that is $C^\infty(D)$ in the whole open disk (including the origin!). Furthermore this u meets the boundary condition in the sense that
$$\lim_{r\uparrow 1} \|u(r,\cdot) - f\|_{L^2(-\pi,\pi)} = 0.$$

[1] This is u in polar coordinates.