The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

10.1. Some Fourier transforms. Compute the following one dimensional Fourier transforms for

$$e^{ix-|x|^2}, \quad e^{-a|x|}, \quad e^{-ax}\sin(3x)\mathbf{1}_{(0,\infty)},$$

where $a \in \mathbb{C}, \mathrm{Re}(a) > 0$.

(BONUS) Given $u, v \in \mathcal{S}(\mathbb{R})$ compute the Fourier transforms of

$$(x, y) \mapsto u(2x)v(y/2)$$

in terms of $\hat{u}, \hat{v}$.

10.2. Dominated convergence review. Motivate each of the following statements using the dominated convergence theorem in a suitable measure space, but pay attention: one of them is in fact *false*!

1. Given $f \in L^2(\mathbb{R}^d)$ it holds

$$\int_{\{|x|>R\}} f(x)^2 \sin(x_1)\, dx \to 0 \text{ as } R \to \infty$$

2. Given $f \in L^1(\mathbb{R}^d)$ it holds

$$\int_{\{|x|>R\}} \frac{\sqrt{1+f(x)^2}-1}{1+\hat{f}(x)}\, dx \to 0 \text{ as } R \to \infty$$

3. Let $\psi \in C^1_c(\mathbb{R}^d)$ such that $\psi(x) \equiv 1$ in a neighbourhood of $x = 0$. Then for each $f \in L^1(\mathbb{R}^d)$ it holds

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^d} f(x)\psi(\epsilon x)\, dx = \int_{\mathbb{R}^d} f(x)\, dx \quad \text{and} \quad \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} f(x)\partial_{x_j}\psi(\epsilon x)\, dx = 0.$$

4. Let $\{c_k\} \in \ell^1(\mathbb{N})$. The map $f(t) := \sum_{k\in\mathbb{N}} c_k e^{i\sin(k)t}$ is of class $C^\infty(\mathbb{R})$ and its derivatives are given by $f^{(m)}(t) = \sum_{k\in\mathbb{N}}(i\sin(k))^m c_k e^{i\sin(k)t}$.

5. Let $\{c_k\} \in \ell^2(\mathbb{N})$. The map $f(t) := \sum_{k\in\mathbb{N}} c_k e^{ikt^2}$ is of class $C^1(\mathbb{R})$ and its derivative is given by $f'(t) = 2it\sum_{k\in\mathbb{N}} kc_k e^{ikt^2}$.

10.3. Harmonic functions on the disk. In this problem we show the existence of the so-called harmonic extension in the interior of the disk of a sufficiently regular function f defined on the disk boundary.

Consider the second order differential operators in two variables (x_1, x_2):

$$\Delta := \partial_{11} + \partial_{22} \qquad \text{and} \qquad L := \partial_{11} + \frac{1}{x_1}\partial_1 + \frac{1}{x_1^2}\partial_{22}.$$

We say that a twice differentiable function $w(x_1, x_2)$ is harmonic $\Delta w = 0$ in its domain.

1. Given $u\colon \overline{D} \to \mathbb{R}$, where $D := \{(x, y) : x^2 + y^2 < 1\}$, consider the function
$$v(r, \theta) := u(r\cos\theta, r\sin\theta), \quad r \in [0, 1], \theta \in \mathbb{R}.^1 \tag{1}$$
Using the chain rule, check that $(\Delta u)(r\cos\theta, r\sin\theta) = Lv(r, \theta)$ for all $r \in (0, 1)$ and $\theta \in \mathbb{R}$.

2. Given any regular function $F\colon \partial D \to \mathbb{R}$ consider its 2π periodic version $f\colon \mathbb{R} \to \mathbb{R}$ defined by
$$F(\cos\theta, \sin\theta) =: f(\theta), \quad \theta \in \mathbb{R}.$$
Show that we can find a solution of
$$\begin{cases} \Delta u = 0 & \text{in } D \setminus \{0\}, \\ u = F & \text{on } \partial D, \end{cases}$$
solving instead
$$\begin{cases} \partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v = 0 & \text{in } (0, 1] \times \mathbb{R}, \\ v(r, \theta + 2\pi) = v(r, \theta) & \text{in } (0, 1] \times \mathbb{R}, \\ v(1, \theta) = f(\theta) & \text{for all } \theta \in \mathbb{R}, \end{cases} \tag{2}$$
and then defining u trough (1).

3. Formally solve the system (2) by the Ansatz $v := \sum_{k\in\mathbb{Z}} u_k(r) e^{ik\theta}$. Explain why the $\{u_k(r)\}$ are not *uniquely* determined by the $\{c_k(f)\}$. Explain why they are unique if we further require that
$$\limsup_{r\downarrow 0} |u_k(r)| < \infty \quad \forall k \in \mathbb{Z}. \tag{3}$$

4. Let $v(r, \theta)$ be the Ansatz constructed in the previous point using the extra assumption (3). Show that v is of class C^∞ in the (r, θ) variables in $[0, 1) \times \mathbb{R}$, as soon as $f \in L^2$.

5. $(*)$ show that, as soon as $f \in L^2(-\pi, \pi)$, the v you constructed with the extra assumption (3), corresponds in fact to a u that is $C^\infty(D)$ in the whole open disk (including the origin!). Furthermore this u meets the boundary condition in the sense that
$$\lim_{r\uparrow 1} \|u(r, \cdot) - f\|_{L^2(-\pi,\pi)} = 0.$$

[1] This is u in polar coordinates.

10. Solutions

Solution of 10.1:

1. Set $f(x) = e^{ix-|x|^2}$. Define the operators
$$\tau_\alpha(f) := f(\cdot - \alpha), \quad m_{e^{i\alpha\cdot}}(f) := e^{i\alpha\cdot} f, \quad \delta_\alpha(f) := f(\alpha\cdot),$$
where $\alpha > 0$. Then, we have
$$\mathcal{F} \circ \delta_\alpha = |\alpha|^{-d}\delta_{1/\alpha} \circ \mathcal{F}, \quad \mathcal{F} \circ m_{e^{i\alpha\cdot}} = \tau_\alpha \circ \mathcal{F}.$$
Furthermore, if $\Phi_1(x) := (2\pi)^{-1/2}e^{-x^2/2}$, then $\hat{\Phi}_1 = \Phi_1$. Hence our f can be written as
$$f = \sqrt{2\pi}(m_{e^{i\cdot}} \circ \delta_{\sqrt{2}})(\Phi_1),$$
so taking the Fourier transform of both sides
$$\begin{aligned}\hat{f} &= \sqrt{2\pi}(\mathcal{F} \circ m_{e^{i\cdot}} \circ \delta_{\sqrt{2}})(\Phi_1) \\ &= \sqrt{2\pi}(\tau_1 \circ \mathcal{F} \circ \delta_{\sqrt{2}})(\Phi_1) \\ &= \sqrt{\pi}(\tau_1 \circ \delta_{1/\sqrt{2}} \circ \mathcal{F})(\Phi_1) \\ &= \sqrt{\pi}(\tau_1 \circ \delta_{1/\sqrt{2}})(\Phi_1) = \sqrt{\pi}\Phi_1\Big(\frac{\cdot - 1}{\sqrt{2}}\Big) = \frac{1}{\sqrt{2}}e^{-(\cdot-1)^2/4}.\end{aligned}$$

2. Set $f(x) = e^{-a|x|}$. The function is summable because $|f(x)| = e^{-\mathrm{Re}(a)x}$and $\mathrm{Re}(a) > 0$. Then,
$$\hat{f}(\xi) = \sqrt{\tfrac{1}{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx = \sqrt{\tfrac{1}{2\pi}}\int_{-\infty}^{0} e^{ax}e^{-i\xi x}\,dx + \sqrt{\tfrac{1}{2\pi}}\int_{0}^{\infty} e^{-ax}e^{-i\xi x}\,dx.$$
Hence by direct integration, $\hat{f}(\xi) = \sqrt{\frac{1}{2\pi}}(\frac{1}{a-i\xi} + \frac{1}{a+i\xi}) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}$ which is continuous and vanishes at infinity.

3. Set $f(x) = e^{-ax}\sin(3x)\mathbf{1}_{(0,\infty)}$. Then, using $2i\sin(3x) = e^{3ix} - e^{-3ix}$, we find
$$\hat{f}(\xi) = \sqrt{\tfrac{1}{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx = \tfrac{1}{2i}\sqrt{\tfrac{1}{2\pi}}\int_{0}^{\infty} e^{-(a+(\xi-3)i)x} - e^{-(a-(\xi-3)i)x}\,dx.$$
Hence, $\hat{f}(\xi) = \frac{1}{2i}\sqrt{\frac{1}{2\pi}}(\frac{1}{a+(\xi-3)i} + \frac{1}{a-(\xi-3)i})$.

4. Given $u, v \in \mathcal{S}(\mathbb{R})$, set $g(u, v) = u(2x)v(y/2)$. Then,
$$\begin{aligned}\hat{g}(\xi, \eta) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(2x)v(y/2)e^{-i\xi x}e^{-i\eta y}\,dx\,dy \\ &= 2\cdot(\tfrac{1}{2})\tfrac{1}{2\pi}\int_{-\infty}^{\infty} u(x)e^{-i\xi x/2}\,dx \int_{-\infty}^{\infty} v(y)e^{-i2\eta y}\,dy,\end{aligned}$$
where the second equality is because we could interchange integration order for functions in the Schwartz class and Fubini theorem applies. Hence,
$$\hat{g}(\xi, \eta) = \hat{u}(\xi/2)\hat{v}(2\eta).$$

Solution of 10.2:

1. Consider the family of functions $f_R : \mathbb{R}^d \to \mathbb{R}$ given by $f_R(x) = \mathbf{1}_{B_R^c}(x) \cdot f(x)^2 \sin(x_1)$. We have pointwise $f_R \to 0$ as $R \to \infty$. Moreover, for every $x \in \mathbb{R}^d, |f_R(x)| \leq |f(x)|^2 \cdot |\sin(x_1)| \leq |f(x)|^2$. Note that $f \in L^2(\mathbb{R}^d)$ implies that $\int_{\mathbb{R}^d} |f(x)|^2 dx < \infty$, that is, $|f|^2$ is summable. Thus the family f_R is pointwise convergent and pointwise bounded by a summable function, so we can apply the dominated convergence theorem in $L^1(\mathbb{R}^d)$ to conclude
$$\int_{\{|x|>R\}} f(x)^2 \sin(x_1)\, dx = \int_{\mathbb{R}^d} f_R(x)\, dx \to \int_{\mathbb{R}^d} 0\, dx = 0 \text{ as } R \to \infty.$$

2. Consider the family of functions $f_R : \mathbb{R}^d \to \mathbb{R}$, given by $f_R(x) = \mathbf{1}_{B_R^c} \frac{\sqrt{1+f(x)^2}-1}{1+\hat{f}(x)}$. Since f is summable, we know that its Fourier transform $\hat{f}$ approaches zero as $|x| \to \infty$. Thus there exists some $M > 0$, such that for all x with $|x| \geq M$, it holds that $|1 + \hat{f}(x)| \geq \frac{1}{2}$. By definition of f_R we have $f_R \to 0$ pointwise and for every $R > M$ and $x \in \mathbb{R}^d$, we have
$$|f_R(x)| \leq \frac{|\sqrt{1+f(x)^2}-1|}{1/2} \leq 2|f(x)|,$$
since $\sqrt{1+f(x)^2} - 1 \leq \sqrt{f(x)^2}$. So f_R is pointwise convergent and dominated (for R big enough) by a summable function. Hence dominated convergence for $L^1(\mathbb{R}^d)$ gives
$$\int_{\{|x|>R\}} \frac{\sqrt{1+f(x)^2}-1}{1+\hat{f}(x)}\, dx = \int_{\mathbb{R}^d} f_R(x)\, dx \to 0 \text{ as } R \to \infty.$$

3. Consider the family of functions $f_\varepsilon : \mathbb{R}^d \to \mathbb{R}$, given by $f_\varepsilon(x) = f(x)\psi(\varepsilon x)$. We know that $\psi \equiv 1$ in $B_r(0)$ for some $r > 0$. Then for any $x \in B_{r/\varepsilon}(0)$, εx is in $B_r(0)$ and so $f_\varepsilon(x) = f(x)\psi(\varepsilon x) = f(x)$. This shows that $f_\varepsilon \to f$ pointwise as $\varepsilon \to 0$. Since $\psi \in C_c^1(\mathbb{R}^d)$, it is bounded, $|f_\varepsilon(x)| \leq ||\psi||_{L^\infty}|f(x)|$ and so $||\psi||_{L^\infty}|f|$ is summable and bounds f_ε pointwise. We can apply the dominated convergence theorem in $L^1(\mathbb{R}^d)$ and get
$$\int_{\mathbb{R}^d} f(x)\psi(\varepsilon x)\, dx = \int_{\mathbb{R}^d} f_\varepsilon(x)\, dx \to \int_{\mathbb{R}^d} f(x)\, dx \text{ as } \varepsilon \to 0.$$

For the second limit, notice $\psi \in C_c^1(\mathbb{R}^d)$ implies that also its derivatives are compactly supported, i.e. $\partial_{x_j}\psi \in C_c^0(\mathbb{R}^d)$ and so they are bounded as well. Moreover, since ψ is constant in a neighbourhood of 0, there is some $r > 0$ such that $\partial_{x_j}\psi \equiv 0$ in $B_r(0)$. Thus by the same argument as above, the family $g_\varepsilon(x) = f(x)\partial_{x_j}\psi(\varepsilon x)$ converges pointwise to 0 as $\varepsilon \to 0$. By boundedness, $|g_\varepsilon(x)| \leq ||\partial_{x_j}\psi||_{L^\infty}|f(x)|$, where the upper bound is again summable. So again by dominated convergence in $L^1(\mathbb{R}^d)$:
$$\int_{\mathbb{R}^d} f(x)\partial_{x_j}\psi(\varepsilon x)\, dx = \int_{\mathbb{R}^d} g_\varepsilon(x)\, dx \to 0 \text{ as } \varepsilon \to 0.$$

4. Notice that for any $t \in \mathbb{R}$, $|f(t)| \leq \sum_k |c_k e^{i\sin(k)t}| = \sum_k |c_k| < \infty$ and thus f is a well-defined function. We first show that f is continuous: For any $k \in \mathbb{N}$, we have $c_k e^{i\sin(k)s} \to c_k e^{i\sin(k)t}$ as $s \to t$. Moreover, $|c_k e^{i\sin(k)s}| \leq |c_k|$, which is summable. By the dominated convergence theorem for $L^1(\mathbb{N}, \mathcal{P}(\mathbb{N}), \#)$ we have:

$$f(s) = \sum_{k\in\mathbb{N}} c_k e^{i\sin(k)s} \to \sum_{k\in\mathbb{N}} c_k e^{i\sin(k)t} = f(t) \text{ as } s \to t,$$

which shows continuity of f. Next, we want to compute the derivative $f'(t)$. For this, consider $\frac{f(t+h)-f(t)}{h} = \sum_k \frac{1}{h}\left(e^{i\sin(k)h} - 1\right) c_k e^{i\sin(k)t}$. For every $k \in \mathbb{N}$, $\frac{1}{h}\left(e^{i\sin(k)h} - 1\right) \cdot c_k e^{i\sin(k)t} \to i\sin(k) \cdot c_k e^{i\sin(k)t}$ as $h \to 0$ and for all $h \in \mathbb{R}$, $|\frac{1}{h}\left(e^{i\sin(k)h} - 1\right) \cdot c_k e^{i\sin(k)t}| \leq 2|\sin(k)| \cdot |c_k e^{i\sin(k)t}| \leq 2|c_k|$, which is summable. So we can invoke again dominated convergence for $L^1(\mathbb{N}, \mathcal{P}(\mathbb{N}), \#)$ to conclude that

$$\begin{aligned} f'(t) &= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \\ &= \lim_{h\to 0} \sum_{k\in\mathbb{N}} c_k \frac{1}{h}\left(e^{i\sin(k)h} - 1\right) e^{i\sin(k)t} \\ &= \sum_{k\in\mathbb{N}} \lim_{h\to 0} c_k \frac{1}{h}\left(e^{i\sin(k)h} - 1\right) e^{i\sin(k)t} = \sum_{k\in\mathbb{N}} i\sin(k) c_k e^{i\sin(k)t}. \end{aligned}$$

Similar to the argument above, we can show continuity of f', and thus $f \in C^1(\mathbb{R})$. We obtain $f \in C^\infty(\mathbb{R})$ and the identity $f^{(m)}(t) = \sum_k (i\sin(k))^m c_k e^{i\sin(k)t}$ by iterating the exact same steps above and c_k by $(i\sin(k))^m c_k$.

5. This statement is actually false: Consider the sequence $\{\frac{1}{k}\}_k \in \ell^2(\mathbb{Z})$ and the map $f(t) = \sum_{k\in\mathbb{N}} \frac{1}{k} e^{ikt^2}$. This sum diverges for $t = 0$ and so f is not even continuous at the point $t = 0$.

Solution of 10.3:

1. It is convenient to define $\Phi(r,\theta) = (r\cdot\cos\theta, r\cdot\sin\theta)$. Then $v = u \circ \Phi$. Further, we notice that

$$\begin{cases} \frac{\partial\Phi_1}{\partial r} = \cos\theta & \frac{\partial\Phi_1}{\partial\theta} = -r\sin\theta \\ \frac{\partial\Phi_2}{\partial r} = \sin\theta & \frac{\partial\Phi_2}{\partial\theta} = r\cos\theta. \end{cases}$$

Using this and the chain rule, we obtain

$$\begin{aligned} \partial_r v &= \partial_1 u \cdot \frac{\partial\Phi_1}{\partial r} + \partial_2 u \cdot \frac{\partial\Phi_2}{\partial r} \\ &= \cos\theta \cdot (\partial_1 u)\circ\Phi + \sin\theta\cdot(\partial_2 u)\circ\Phi. \\ \partial_\theta v &= \partial_1 u \cdot \frac{\partial\Phi_1}{\partial\theta} + \partial_2 u \cdot \frac{\partial\Phi_2}{\partial\theta} \\ &= r\cos\theta\cdot(\partial_2 u)\circ\Phi - r\sin\theta\cdot(\partial_1 u)\circ\Phi. \end{aligned}$$

Another application of the chain rule gives us:

$$\begin{aligned}
\partial_{rr} v =& \partial_r(\partial_r v) \\
=& \cos\theta \left(\partial_{11} u \cdot \frac{\partial \Phi_1}{\partial r} + \partial_{12} u \cdot \frac{\partial \Phi_2}{\partial r} \right) \\
& + \sin\theta \cdot \left(\partial_{21} u \cdot \frac{\partial \Phi_1}{\partial r} + \partial_{22} u \cdot \frac{\partial \Phi_2}{\partial r} \right) \\
=& \cos^2\theta \cdot (\partial_{11} u) \circ \Phi + 2\cos\theta\sin\theta(\partial_{12} u) \circ \Phi + \sin^2\theta \cdot (\partial_{22} u) \circ \Phi.
\end{aligned}$$

We can compute $\partial_{\theta\theta} v$ by the same method:

$$\begin{aligned}
\partial_{\theta\theta} v =& \partial_\theta(\partial_\theta v) \\
=& \partial_\theta \left(r\cos\theta \cdot (\partial_2 u) \circ \Phi - r\sin\theta \cdot (\partial_1 u) \circ \Phi \right) \\
=& r\sin\theta \cdot (\partial_2 u) \circ \Phi + r\cos\theta \left(\partial_{21} u \cdot \frac{\partial \Phi_1}{\partial \theta} + \partial_{22} u \cdot \frac{\partial \Phi_2}{\partial \theta} \right) \\
& - r\cos\theta \cdot (\partial_1 u) \circ \Phi - r\sin\theta \cdot \left(\partial_{11} u \cdot \frac{\partial \Phi_1}{\partial \theta} + \partial_{12} u \cdot \frac{\partial \Phi_2}{\partial \theta} \right) \\
=& - r\left(\cos\theta \cdot \partial_1 u \circ \Phi + \sin\theta \cdot \partial_2 u \circ \Phi \right) \\
& + r^2 \left(\sin^2\theta \partial_{11} u \circ \Phi - 2\cos\theta\sin\theta \partial_{12} u \circ \Phi + \cos^2\theta \partial_{22} u \circ \Phi \right) \\
=& - r\partial_r v + r^2 \left(\sin^2\theta \partial_{11} u \circ \Phi - 2\cos\theta\sin\theta \partial_{12} u \circ \Phi + \cos^2\theta \partial_{22} u \circ \Phi \right)
\end{aligned}$$

Note that this implies (by using $\cos^2\theta + \sin^2\theta = 1$) that

$$\frac{1}{r^2}\partial_{\theta\theta} v + \partial_{rr} v = -\frac{1}{r}\partial_r v + \partial_{11} u \circ \Phi + \partial_{22} u \circ \Phi.$$

Now the claim boils down to a direct computation:

$$\begin{aligned}
(\Delta u)(r \cdot \cos\theta, r \cdot \sin\theta) &= (\partial_{11} u) \circ \Phi + (\partial_{22} u) \circ \Phi \\
&= \frac{1}{r^2}\partial_{\theta\theta} v + \partial_{rr} v + \frac{1}{r}\partial_r v = Lv(r,\theta).
\end{aligned}$$

2. Assume that $v : (0,1] \times \mathbb{R} \to \mathbb{R}$ is a solution to

$$\begin{cases}
\partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v = 0 & (r,\theta) \in (0,1] \times \mathbb{R} \\
v(r, \theta + 2\pi) = v(r,\theta) & (r,\theta) \in (0,1] \times \mathbb{R} \\
v(1,\theta) = f(\theta) & \theta \in \mathbb{R}.
\end{cases}$$

Then there must exist $u \colon \bar{D} \setminus \{0\} \to \mathbb{R}$ such that $v(r,\theta) = u(r\cos\theta, r\sin\theta)$. By the first part of the exercise, we have that

$$\begin{aligned}
\Delta u = Lv &= \partial_{rr} v + r^{-1}\partial_r v + r^{-2}\partial_{\theta\theta} v \\
&= r^{-2} \cdot (\partial_{\theta\theta} v + r\partial_r v + r^2 \partial_{rr} v) = 0,
\end{aligned}$$

on $D \setminus \{0\}$ which corresponds to considering $(r,\theta) \in (0,1) \times \mathbb{R}$. The initial condition can also be checked directly: Let $x \in \partial D$, then there is $\theta \in \mathbb{R}$ such that $x = (\cos\theta, \sin\theta)$. This implies

$$u(x) = v(1,\theta) = f(\theta) = F(\cos\theta, \sin\theta) = F(x).$$

3. We now want to solve the PDE from part (2) formally. For this we assume

$$v := \sum_{k\in\mathbb{Z}} u_k(r)e^{ik\theta}.$$

We have

$$\begin{cases} \partial_{\theta\theta} v = -\sum_{k\in\mathbb{Z}} k^2 u_k(r)e^{ik\theta} \\ \partial_r v = \sum_{k\in\mathbb{Z}} u_k'(r)e^{ik\theta} \\ \partial_{rr} v = \sum_{k\in\mathbb{Z}} u_k''(r)e^{ik\theta}. \end{cases}$$

Thus, the PDE $\partial_{\theta\theta}v + r\partial_r v + r^2\partial_{rr}v = 0$ becomes the following system of ODEs:

$$k^2 u_k(r) = ru_k'(r) + r^2 u_k''(r), \qquad \text{for all } k \in \mathbb{Z}.$$

We first consider the case $k = 0$, where we have $r \cdot u_0'(r) + r^2 \cdot u_0''(r) = 0$. This ODE is solved by $u_0(r) = c_0 + d_0 \cdot \log r$, for some arbitrary constants c_0, d_0. For $k \neq 0$ we have $u_k(r) = c_k r^{|k|} + d_k r^{-|k|}$, for arbitrary constants $\{c_k\}$ and $\{d_k\}$.

We can see that the initial datum $v(1,\theta) = f(\theta)$ is not enough to determine the coefficients, as it only tells us that $u_k(1)$ has to agree with $\{c_k(f)\}$, and since the u_k are determined by a second order ODE, this is not sufficient for uniqueness: there is an additional degree of freedom. If we additionally assume,

$$\limsup_{r\downarrow 0} |u_k(r)| < \infty$$

for all $k \in \mathbb{Z}$, this forces the coefficients of the log and of the negative powers of r to be zero. Thus, we obtain $d_k = 0$ for all $k \in \mathbb{Z}$. And matching the condition $u_k(1) = c_k(f)$ we find Thus, $c_k = c_k(f)$ for all $k \in \mathbb{Z}$. We obtain

$$v(r,\theta) = \sum_{k\in\mathbb{Z}} c_k(f) r^{|k|} e^{ik\theta}.$$

4. We now want to show that the solution constructed in the last part is smooth if $f \in L^2$. We know that r^k decays faster than any power of k if $|r| < 1$. Consider now the $\Omega_\varepsilon := [0, 1-\varepsilon) \times \mathbb{R}$. We show that $v|_{\Omega_\varepsilon}$ is smooth for any $\varepsilon > 0$. For this let $\alpha, \beta \in \mathbb{N}$ be arbitrary. Then on Ω_ε we obtain

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \|\partial_r^\alpha \partial_\theta^\beta u_k(r)e^{ik\theta}\|_{L^\infty(\Omega_\varepsilon)} &= \sum_{k\in\mathbb{Z}} \left\|\partial_r^\alpha \partial_\theta^\beta (c_k(f) r^{|k|} e^{ik\theta})\right\|_{L^\infty(\Omega_\varepsilon)} \\ &\le \sum_{k\in\mathbb{Z}} |c_k(f)||k|^\beta \left\|\partial_r^\alpha r^{|k|}\right\|_{L^\infty(\Omega_\varepsilon)} \\ &\le \sum_{k\in\mathbb{Z}} |c_k(f)||k|^{\beta+1}(|k|-1)\dots(|k|-\alpha+1)(1-\varepsilon)^{|k|-\alpha} < \infty, \end{aligned}$$

where we use that $\{c_k(f)\} \in \ell^2 \subset \ell^\infty$ in the end. Thus, the partial sums and all their derivatives converge uniformly on Ω_ε, which implies $v|_{\Omega_\varepsilon} \in C^\infty(\Omega_\varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we have that v is smooth on $[0,1) \times \mathbb{R}$.

5. We first show that v fulfils the boundary condition in the sense that

$$\lim_{r\uparrow 1} \|u(r,\cdot)-f\|_{L^2(-\pi,\pi)} = 0.$$

Note that for any $k \in \mathbb{Z}$ we have

$$c_k(u(r,\cdot)-f) = c_k(f)r^k - c_k(f).$$

We see directly, that $c_k(u(r,\cdot)-f) \to 0$ as $r \uparrow 1$ pointwise. Thus, $\lim_{r\uparrow 1} \|u(r,\cdot)-f\|_{L^2(-\pi,\pi)} = 0$ by Parseval's identity and dominated convergence with dominant

$$|c_k(f)|^2(1-r^k)^2 \leq |c_k(f)|^2 \in \ell^1(\mathbb{Z}).$$

We remark that if we have stronger decay conditions on $c_k(f)$ (i.e., f is more regular) then the boundary datum is achieved in stronger norms. For example, if $\sum_k |c_k(f)| < \infty$, then the same computation shows

$$\lim_{r\downarrow 0} \|u(r,\cdot)-f\|_{L^\infty(-\pi,\pi)} = 0,$$

which means that u can be continuously extended to f on ∂D.

We now want to see that v corresponds to a solution $u \in C^\infty(D)$ of the Laplace equation. The relationship between u and v is given by $v(r,\theta) = u(r\cos\theta, r\sin\theta)$, note that if we show $u \in C^2(D)$ we automatically obtain that u is a solution of $\Delta u = 0$ in the whole D, by continuity.

Recall $\Phi(r,\theta) = (r\cdot\cos\theta, r\cdot\sin\theta)$ from part (1). For $(r,\theta) \neq 0$ this is a local diffeomorphism. Thus we obtain that u can be written around any $(x,y) \in D \setminus \{0\}$ as the composition of smooth functions. Namely of v and a local inverse of Φ. Thus, u is smooth on $D \setminus \{0\}$. We still have to show it at the origin, since the change of variables is singular there we have to work in cartesian coodinates.

In order to do so we express

$$\begin{aligned}
(\partial_1 u)\circ\Phi &= \cos\theta\,\partial_r v - \frac{1}{r}\sin\theta\,\partial_\theta v = \sum_{k\in\mathbb{Z}} c_k(f) r^{|k|-1}(|k|\cos\theta - ik\sin\theta)e^{ik\theta} \\
&= c_1(f)(\cos\theta - i\sin\theta)e^{i\theta} + c_{-1}(f)(\cos\theta + i\sin\theta)e^{-i\theta} + r\tilde{v}(r,\theta) \\
&= c_1(f) + c_{-1}(f) + r\tilde{v}(r,\theta),
\end{aligned}$$

this function is continuous as $r \to 0$, since the dependence on θ is only in the $\tilde{v}$ which is in turn multiplied by r. Let us check first that the same happens for $\partial_2 u$:

$$\begin{aligned}
(\partial_2 u)\circ\Phi &= \sin\theta\,\partial_r v + \frac{1}{r}\cos\theta\,\partial_\theta v = \sum_{k\in\mathbb{Z}} c_k(f) r^{|k|-1}(|k|\sin\theta + ik\cos\theta)e^{ik\theta} \\
&= c_1(f)(\sin\theta + i\cos\theta)e^{i\theta} + c_{-1}(f)(\sin\theta - i\cos\theta)e^{-i\theta} + r\tilde{v}(r,\theta) \\
&= ic_1(f) - ic_{-1}(f) + r\tilde{v}(r,\theta),
\end{aligned}$$

which is again continuous. This little miracle suggests that something is going on. One could prove inductively that this computation works similarly for derivatives of all orders (this is not a surprise, the $\partial_1 u$ solves the same problem with boundary datum $\partial_1 f$, and we did not use the size of the $c_k(f)$ in this computation...), but we present another argument.

Recall that the change of variables Φ can be easily inverted for certain functions, namely

$$(x+iy)^{|k|} = r^{|k|}e^{i|k|\theta}, \quad (x-iy)^{|k|} = r^{|k|}e^{-i|k|\theta},$$

so using this identity and splitting the sum in positive and negative frequencies we can express directly v in cartesian coordinates:

$$u(x,y) = v \circ \Phi^{-1} = c_0(f) + \sum_{k>0} c_k(f)(x+iy)^k + \sum_{k>0} c_{-k}(f)(x-iy)^k,$$

it can be checked by summing the derivatives that this function is C^∞ as long as $x^2+y^2<1$, but we can also remember complex function theory and set $z := x+iy$ and notice that

$$u(x,y) = c_0(f) + \underbrace{\sum_{k>0} c_k(f) z^k}_{:=\phi(z)} + \underbrace{\sum_{k>0} c_{-k}\overline{z}^k}_{=:\psi(z)}.$$

By standard complex analysis $\phi(z)$ is holomorphic in the unit disk and ψ is anti-holomorphic in the unit disk. In particular they both are C^∞.