The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

11.1. Closed answer questions.

1. If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^2(\mathbb{R}^d)$ is it necessarily true that $f \in L^2(\mathbb{R}^d)$?
2. Is the function $\frac{1}{1+ix^4}$ in the Schwartz class $\mathcal{S}(\mathbb{R})$?
3. Show that if $\lambda \in \mathbb{C}$ is an eigenvalue[1] of $\mathcal{F} \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, then necessarily $\lambda \in \{\pm 1, \pm i\}$.
4. Let A be an invertible $d \times d$ matrix with real entries. Compute the Fourier transform of $x \mapsto f(Ax)$ in terms of $\hat{f}$ and A.
5. Given $\psi \in \mathcal{S}(\mathbb{R})$, show that
$$\frac{1}{1+i\xi}\psi(\xi) \in \mathcal{S}(\mathbb{R}).$$
Hint: recall Leibniz formula for higher-order derivatives of products
$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

11.2. Differential operators with constant coefficients. (BONUS) Let $u \in \mathcal{S}(\mathbb{R}^d)$ be a scalar function and $V \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}^d)$ be vector field. Compute the following quantities in terms of $\hat{u}$ and $\hat{V}$[2].

1. $\mathcal{F}(\nabla u)$,
2. $\mathcal{F}(\mathrm{div}\ V)$,
3. $\mathcal{F}(\Delta u)$.

11.3. A differential equation. Given $\phi \in \mathcal{S}(\mathbb{R})$ we consider the differential equation
$$u'(x) + u(x) = \phi(x) \text{ for all } x \in \mathbb{R}.$$

1. Show that there is a unique solution within the class of Schwartz functions.

[1] That is to say: there exists some nonzero function $v \in L^2(\mathbb{R}^d)$ such that $\mathcal{F}v = \lambda v$.
[2] The Fourier of a vector field is taken component-wise, i.e., $\hat{V}(\xi) = (\hat{V}_1(\xi), \dots, \hat{V}_d(\xi))$.

2. Taking the Fourier transform of both sides of the equation, and then the anti-Fourier transform show that
$$u(x) := \int_{\mathbb{R}} a(\xi)\hat{\phi}(\xi)e^{i\xi x}\, d\xi,$$
is indeed a solution of the above problem, for an appropriate function $a(\xi)$ to be determined.

3. Solve again the above ODE, this time with classical methods (multiply by e^t etc..).

4. Check that the two results you found are indeed the same.

11.4. Decay of the Fourier transform and derivatives. Let $f \in L^2(\mathbb{R}^d)$ such that it's Fourier transform decays at infinity as a negative power, i.e., for some $\alpha \geq 0$ and large $M \geq 1$ it holds
$$|\hat{f}(\xi)| \leq M|\xi|^{-\alpha} \text{ for all } |\xi| \geq 1.$$
The goal of this problem is to show that in fact (up to a modification on a zero measure set) $f \in C^k(\mathbb{R}^d)$ for all integers $k < \alpha/2d$.

1. Consider for each $R > 1$ the functions
$$f_R(x) := (2\pi)^{-d/2} \int_{B_R} \hat{f}(\xi)e^{i\xi x}\, d\xi,$$
compute $\hat{f}_R$ and show that $f_R \to f$ in $L^2(\mathbb{R}^d)$.

2. Show that each $f_R \in C^\infty(\mathbb{R}^d)$ but in general $f_R \notin \mathcal{S}(\mathbb{R}^d)$.

3. Using the decay assumption on $\hat{f}$, show that $\{f_R\}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^d)$, provided $\alpha > d$. Conclude that, up to re-definition on a zero measure set, in this case $f \in C(\mathbb{R})$.

4. Applying the same argument to $\partial_{x_j} f_R$, show inductively that $f \in C^k$ whenever $\alpha > d + k$.