The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

**11.1. Closed answer questions.**

1. If $f \in L^1(\mathbb{R}^d)$ and $\hat{f} \in L^2(\mathbb{R}^d)$ is it necessarily true that $f \in L^2(\mathbb{R}^d)$?
2. Is the function $\frac{1}{1+ix^4}$ in the Schwartz class $\mathcal{S}(\mathbb{R})$?
3. Show that if $\lambda \in \mathbb{C}$ is an eigenvalue[1] of $\mathcal{F}\colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, then necessarily $\lambda \in \{\pm 1, \pm i\}$.
4. Let $A$ be an invertible $d \times d$ matrix with real entries. Compute the Fourier transform of $x \mapsto f(Ax)$ in terms of $\hat{f}$ and $A$.
5. Given $\psi \in \mathcal{S}(\mathbb{R})$, show that
$$\frac{1}{1+i\xi}\psi(\xi) \in \mathcal{S}(\mathbb{R}).$$
**Hint**: recall Leibniz formula for higher-order derivatives of products
$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}.$$

**11.2. Differential operators with constant coefficients.** (BONUS) Let $u \in \mathcal{S}(\mathbb{R}^d)$ be a scalar function and $V \in \mathcal{S}(\mathbb{R}^d, \mathbb{R}^d)$ be vector field. Compute the following quantities in terms of $\hat{u}$ and $\hat{V}$[2].

1. $\mathcal{F}(\nabla u)$,
2. $\mathcal{F}(\operatorname{div} V)$,
3. $\mathcal{F}(\Delta u)$.

**11.3. A differential equation.** Given $\phi \in \mathcal{S}(\mathbb{R})$ we consider the differential equation
$$u'(x) + u(x) = \phi(x) \text{ for all } x \in \mathbb{R}.$$

1. Show that there is a unique solution within the class of Schwartz functions.

[1] That is to say: there exists some nonzero function $v \in L^2(\mathbb{R}^d)$ such that $\mathcal{F}v = \lambda v$.
[2] The Fourier of a vector field is taken component-wise, i.e., $\hat{V}(\xi) = (\hat{V}_1(\xi), \ldots, \hat{V}_d(\xi))$.

2. Taking the Fourier transform of both sides of the equation, and then the anti-Fourier transform show that
$$u(x) := \int_{\mathbb{R}} a(\xi)\hat{\phi}(\xi)e^{i\xi x}\, d\xi,$$
is indeed a solution of the above problem, for an appropriate function $a(\xi)$ to be determined.

3. Solve again the above ODE, this time with classical methods (multiply by $e^t$ etc..).

4. Check that the two results you found are indeed the same.

**11.4. Decay of the Fourier transform and derivatives.** Let $f \in L^2(\mathbb{R}^d)$ such that it's Fourier transform decays at infinity as a negative power, i.e., for some $\alpha \geq 0$ and large $M \geq 1$ it holds
$$|\hat{f}(\xi)| \leq M|\xi|^{-\alpha} \text{ for all } |\xi| \geq 1.$$
The goal of this problem is to show that in fact (up to a modification on a zero measure set) $f \in C^k(\mathbb{R}^d)$ for all integers $k < \alpha/2d$.

1. Consider for each $R > 1$ the functions
$$f_R(x) := (2\pi)^{-d/2} \int_{B_R} \hat{f}(\xi)e^{i\xi x}\, d\xi,$$
compute $\hat{f}_R$ and show that $f_R \to f$ in $L^2(\mathbb{R}^d)$.

2. Show that each $f_R \in C^\infty(\mathbb{R}^d)$ but in general $f_R \notin \mathcal{S}(\mathbb{R}^d)$.

3. Using the decay assumption on $\hat{f}$, show that $\{f_R\}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^d)$, provided $\alpha > d$. Conclude that, up to re-definition on a zero measure set, in this case $f \in C(\mathbb{R})$.

4. Applying the same argument to $\partial_{x_j} f_R$, show inductively that $f \in C^k$ whenever $\alpha > d + k$.

# 11. Solutions

**Solution of 11.1:**

1. Yes this is true. As the (inverse) Fourier transform is an isometry in $L^2$ (Plancherel's formula) and so since

$$\|f\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\hat{f}\|_{L^2(\mathbb{R}^d)} = \|\hat{f}\|_{L^2(\mathbb{R}^d)} < \infty.$$

2. Let $f(x) = 1/(1+ix^4)$. For $f$ to belong to the Schwartz space $\mathcal{S}(\mathbb{R})$, one must have $x \mapsto x^n D^m f \in L^\infty(\mathbb{R})$, for all $n, m \in \mathbb{N}$. However, if we choose $m = 0$ and $n > 4$, it is clear that

$$\frac{|x^n|}{|1+ix^4|} = \frac{|x|^n}{(1+x^8)^{1/2}} \notin L^\infty(\mathbb{R}),$$

hence $f$ does not belong to $\mathcal{S}(\mathbb{R})$.

3. Assume $\lambda \in \mathbb{C}$ is an eigenvalue of the Fourier transform and let $f \in L^2(\mathbb{R}^d)$ be its associated eigenvector, i.e. $\mathcal{F}(f) = \lambda f$. We know that the Fourier transform is an isometry on $L^2$, with inverse the inverse Fourier transform. We note that $\mathcal{F}(f)(x) = \mathcal{F}^{-1}(f)(-x)$, for all $f \in L^2, x \in \mathbb{R}^d$. Thus, one has $\mathcal{F}^2(f)(x) = f(-x)$, for all $x \in \mathbb{R}^d$ and hence $\mathcal{F}^4(f) = f$, for all $f \in L^2$. Thus, our eigenvalue $\lambda$ must satisfy $\lambda^4 = 1$, which implies $\lambda \in \{\pm 1, \pm i\}$.

4. Let $f \in L^1(\mathbb{R}^d)$, define $\tilde{f}(x) := f(Ax)$, where $A$ is an invertible $d \times d$ matrix with real entries. Since $A$ is invertible we have $\tilde{f} \in L^1(\mathbb{R}^d)$. We compute $\mathcal{F}(\tilde{f})$.

$$\begin{aligned}
\mathcal{F}(\tilde{f})(\xi) &= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(Ax)\, e^{-ix\cdot\xi}\, dx \\
&\overset{(y=Ax)}{=} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{-i(A^{-1}y)\cdot\xi} \left|\det(A^{-1})\right| dy \\
&= \frac{1}{|\det(A)|(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(y) e^{-iy\cdot\left((A^{-1})^T\xi\right)} dy \\
&= \frac{1}{|\det(A)|} \mathcal{F}(f)\Big((A^{-1})^T\xi\Big).
\end{aligned}$$

5. Take any differential and polynomial order $\alpha, \beta \geq 0$ and estimate the supremum norm in $\mathbb{R}$

$$\begin{aligned}
\left\|x^\beta \partial^\alpha \frac{1}{1+xi}\psi(x)\right\|_\infty &= \left\|x^\beta \sum_{k=0}^{\alpha} \binom{\alpha}{k} \partial^k \frac{1}{1+xi}\, \partial^{\alpha-k}\psi(x)\right\|_\infty \\
&\leq \sum_{k=0}^{\alpha} \binom{\alpha}{k} \left\|x^\beta \frac{(-i)^k k!}{(1+xi)^{k+1}}\, \partial^{\alpha-k}\psi(x)\right\|_\infty \\
&= \sum_{k=0}^{\alpha} \binom{\alpha}{k} k! \underbrace{\left\|x^\beta \partial^{\alpha-k}\psi(x)\right\|_\infty}_{<\infty!} \\
&< +\infty.
\end{aligned}$$

**Solution of 11.2:** By direct computation

1. $\mathcal{F}(\nabla u) = i\xi\hat{u}(\xi)$,
2. $\mathcal{F}(\text{div } V) = i\xi \cdot \hat{V}(\xi)$,
3. $\mathcal{F}(\Delta u) = -|\xi|^2\hat{u}(\xi)$.

**Solution of 11.3:**

1. Suppose that $u, v \in \mathcal{S}(\mathbb{R})$ are solutions to the differential equation. We then define the difference $w := u - v \in \mathcal{S}(\mathbb{R})$ and note it solves the following differential equation

$$w' + w = (u-v)' + (u-v) = u' + u - (v' - v) = \phi - \phi = 0.$$

This is a first order linear differential equation with the solution $w(t) = ce^{-t}$ for any $c \in \mathbb{C}$. Since $w \in \mathcal{S}(\mathbb{R})$ we must have that $w$ is bounded, which can only be the case if $c = 0$. From this we infer the uniqueness since $w \equiv 0$ means $u = v$.

2. Since both sides of the equation are $L^1$ functions, we may take the Fourier transform. We compute

$$\mathcal{F}\phi(\xi) = \mathcal{F}(u + u')(\xi) = \mathcal{F}u(\xi) + i\xi\mathcal{F}u(\xi) = (1 + i\xi)\mathcal{F}u(\xi).$$

Dividing by $1 + i\xi$ (which is never zero!) yields

$$\mathcal{F}u(\xi) = \frac{1}{1+i\xi}\mathcal{F}\phi(\xi).$$

Since $\phi \in \mathcal{S}(\mathbb{R})$ we know that (Theorem 3.25) $\mathcal{F}\phi \in \mathcal{S}(\mathbb{R})$. By Exercise 11.1.5 $\mathcal{F}u \in \mathcal{S}(\mathbb{R})$, which by Theorem 3.25 will mean that $u$ is itself a Schwartz function. As a consequence we are able to apply the inverse Fourier transform to get

$$u(x) = \mathcal{F}^{-1}\mathcal{F}u(x) = \mathcal{F}^{-1}\frac{1}{1+i\xi}\mathcal{F}\phi(x) = \int_{\mathbb{R}} \frac{1}{1+i\xi}\mathcal{F}\phi(\xi)e^{i\xi x}\, d\xi.$$

So the claim holds if we define the integral coefficient

$$a(\xi) := \frac{1}{1+\xi i}.$$

3. Multiply the differential equation by $e^x$ to get

$$e^x\phi(x) = u(x)e^x + u'(x)e^x = (u(x)e^x)'.$$

Since $e^x\phi(x) \to 0$ as $x \to -\infty$ ($\phi$ is bounded), we can take the integral on the interval $(-\infty, x)$ to get

$$u(x)e^x = \int_{-\infty}^x e^t\phi(t)\, dt.$$

Finally divide by $e^x$, which yields the solution

$$u(x) = \int_{-\infty}^x e^{t-x}\phi(t)\, dt.$$

4. We have found two smooth solutions:

$$u_1(x) = \mathcal{F}^{-1}\frac{1}{1+i\xi}\mathcal{F}\phi \quad \text{and} \quad u_2(x) = \int_{-\infty}^{x} e^{t-x}\phi(t)\,dt,$$

we wish to show that $u_1 \equiv u_2$. Notice that if we introduce the $L^1(\mathbb{R})$ function

$$h(x) := e^{-x}\mathbf{1}_{(0,\infty)}(x),$$

then $u_2$ can be written as a convolution, namely

$$u_2(x) = \int_{\mathbb{R}} e^{t-x}\mathbf{1}_{(0,\infty)}(x-t)\phi(t)\,dt = (h*\phi)(x),$$

thus by the properties of the Fourier transform we find (notice that both $h$ and $\phi$ are in $L^1(\mathbb{R}^d)$)

$$\hat{u}_2(\xi) = \sqrt{2\pi}\hat{h}(\xi)\hat{\phi}(\xi) = \frac{1}{1+i\xi}\hat{\phi}(\xi) = \hat{u}_1(\xi), \text{ for all } \xi \in \mathbb{R},$$

since the Fourier transform is injective we conclude $u_1 \equiv u_2$. We used that $\hat{h}(\xi) = \frac{1}{\sqrt{2\pi}}\frac{1}{1+i\xi}$.

**Solution of 11.4:**

1. This is a reality check. Using the fact that the Fourier Transform is an isometry of $L^2$ and $\widehat{f}\,\chi_{B_R(0)} \in L^2(\mathbb{R}^d)$ we get

$$\widehat{f_R} = \mathcal{F}(\mathcal{F}^{-1}(\widehat{f}\,\chi_{B_R(0)})) = \widehat{f}\,\chi_{B_R(0)}.$$

This allows to compute

$$\|f - f_R\|_{L^2}^2 = \left\|\widehat{f} - \widehat{f}_R\right\|_{L^2}^2 = \int_{\mathbb{R}^d}\left|\widehat{f}(\xi)\right|^2 |1-\chi_{B_R(0)}(\xi)|^2\,d\xi \longrightarrow 0,$$

as $R \to \infty$. Here we used dominated convergence with

$$\left|\widehat{f}(\xi)\right|^2\ |1-\chi_{B_R(0)}(\xi)|^2 \le \left|\widehat{f}(\xi)\right|^2 \in L^1(\mathbb{R}^d).$$

2. Note that by the computation above $\operatorname{supp}(\widehat{f_R}) \subset B_R(0)$ so $\widehat{f_R} \in L^1 \cap L^2$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ with

$$\psi \equiv 1 \text{ on } B_R(0).$$

Then

$$f_R = \mathcal{F}^{-1}\left(\widehat{f_R}\right) = \mathcal{F}^{-1}\left(\widehat{f_R}\,\psi\right) = (2\pi)^{d/2} f_R * \mathcal{F}^{-1}(\psi).$$

Since $\psi \in \mathcal{S}(\mathbb{R}^d)$ also $\mathcal{F}^{-1}(\psi) \in \mathcal{S}(\mathbb{R}^d)$ and by standard properties of convolutions we get $f_R \in C^\infty(\mathbb{R}^d)$. For the second part assume that $f_R \in \mathcal{S}(\mathbb{R}^d)$, then also $\widehat{f_R} \in \mathcal{S}(\mathbb{R}^d)$. But $\widehat{f_R}$ might not even be continuous. Take for example $f(x) = (2\pi)^{-d/2}\exp(-|x|^2/2)$ with $\widehat{f} = f$ and

$$\widehat{f_R} = \widehat{f}\,\chi_{B_R(0)} \notin C^0(\mathbb{R}^d).$$

3. Assume that $\alpha > d$. Observe that by $\widehat{f}_R \in L^1(\mathbb{R}^d)$, since it has compact support and $\hat{f} \in L^2$. Let $1 < R_1 \leq R_2 < \infty$ and get

$$\begin{aligned} \|f_{R_2} - f_{R_1}\|_{L^\infty} &\leq (2\pi)^{-d/2} \|\widehat{f}_{R_2} - \widehat{f}_{R_1}\|_{L^1} \\ &\leq (2\pi)^{-d/2} \int_{B_{R_2}(0) \setminus B_{R_1}(0)} |\widehat{f}(\xi)| \, d\xi \\ &\leq (2\pi)^{-d/2} M \int_{\{|\xi| \geq R_1\}} |\xi|^{-\alpha} \, d\xi \longrightarrow 0, \end{aligned}$$

as $R_1 \to \infty$. Here we used dominated convergence with

$$|\xi|^{-\alpha} \, \chi_{B_{R_1}(0)^c}(\xi) \leq |\xi|^{-\alpha} \, \chi_{B_1(0)^c}(\xi) \in L^1(\mathbb{R}^d), \text{ since } \alpha > d.$$

This shows that $(f_R)_{R>1} \subset C^0(\mathbb{R}^d)$ is a Cauchy sequence and since $C^0(\mathbb{R}^d)$ is a Banach space we get $f_R \to g \in C^0(\mathbb{R}^d)$ and pointwise. On the other hand we also know $f_R \to f$ in $L^2(\mathbb{R}^d)$ and (up to a subsequence) pointwise a.e., so we must in fact have $g = f$ up to a redefinition on a null set.

4. Assume that $\alpha > d + k$. Recall the multiindex notation and let $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq k$. Using the version of Proposition 3.15 for the inverse Fourier transform we obtain

$$\partial^\beta f_R = \mathcal{F}^{-1}(i \, \xi^\beta \widehat{f}_R),$$

where we used the facts that $\widehat{f}_R(\xi) \in L^1(\mathbb{R}^d)$ and $\xi^\beta \widehat{f}_R(\xi) \in L^1(\mathbb{R}^d)$. Let $1 < R_1 \leq R_2 < \infty$ and get

$$\begin{aligned} \|\partial^\beta f_{R_2} - \partial^\beta f_{R_1}\|_{L^\infty} &\leq (2\pi)^{-d/2} \left\| i \, \xi^\beta \left( \widehat{f}_{R_2}(\xi) - \widehat{f}_{R_1}(\xi) \right) \right\|_{L^1} \\ &\leq (2\pi)^{-d/2} \int_{B_{R_2}(0) \setminus B_{R_1}(0)} |\widehat{f}(\xi)| \, |\xi|^{|\beta|} \, d\xi \\ &\leq (2\pi)^{-d/2} M \int_{\{|\xi| \geq R_1\}} |\xi|^{-(\alpha - k)} \, d\xi \longrightarrow 0, \end{aligned}$$

as $R_1 \to \infty$. Here we used dominated convergence with

$$|\xi|^{-(\alpha-k)} \, \chi_{B_{R_1}(0)^c}(\xi) \leq |\xi|^{-(\alpha-k)} \, \chi_{B_1(0)^c}(\xi) \in L^1(\mathbb{R}^d).$$

This shows that $(f_R)_{R>1} \subset C_b^k(\mathbb{R}^d)$ is a Cauchy sequence and since $C_b^k(\mathbb{R}^d)$ is a Banach space we get for each multi-index $|\beta| \leq k$ that

$$\partial^\beta f_R \to g_\beta \in C_b(\mathbb{R}^d),$$

by Analysis II we know that we must have

$$\partial^\beta g_0 = g_\beta \text{ for all multi-indeces} |\beta| \leq k.$$

Furthermore by uniqueness of the $L^2$ limit, arguing as above, we also have $g_0 = f$ a.e.. Thus we proved that, up to modifying $f$ in a zero measure set we have $f \in C_b^k(\mathbb{R}^d)$.