The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

12.1. Closed answer questions.

1. Let (X, d) be a metric space and let $q \in X$. Assume you have a sequence $\{x_k\} \subset X$ such that *any* subsequence $\{x_{k_j}\}$ must possess a sub-subsequence $\{x_{j_{k_\ell}}\}$ such that $x_{j_{k_\ell}} \to q$ as $\ell \to \infty$. Is it true that $x_k \to q$ as $k \to \infty$?
2. Let $u \in \mathcal{S}(\mathbb{R})$, show that $\|\hat{u}\|^4_{L^4(\mathbb{R})} = 2\pi \|u * u\|^2_{L^2(\mathbb{R})}$.
3. Consider the set
$$X := \{u \in L^2(\mathbb{R}^d) : \hat{u}(\xi) \equiv 0 \text{ for almost every } |\xi| > 2\},$$
show that X is closed and not open in $L^2(\mathbb{R}^d)$. **Hint**: First show that $\mathcal{F}(X)$ is closed. Then...?
4. Consider the set
$$X := \{u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) : \int_{\mathbb{R}} u = 0\},$$
show that X is dense in $L^2(\mathbb{R})$. What happens if we consider the exact same question in $[-\pi, \pi]$ instead of $\mathbb{R}$? **Hint**: again, try computing $\mathcal{F}(X)$ and use that $\mathcal{F}$ is an isometry.
5. Let H be an Hilbert space and $T : H \to H$ a linear operator of which we know the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ and the corresponding eigenfunctions $\{v_k\}_{k\in\mathbb{N}}$. Suppose that there exists an operator S such that $S \circ T = T \circ S = \mathrm{Id}$. Show that $\lambda_k \neq 0$ for every $k \in \mathbb{N}$ and determine the set of eigenvalues of S.

12.2. A simple compact operator on ℓ^2. Consider the map $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by
$$T((x_k)_{k\in\mathbb{N}}) = \left(\frac{x_k}{k^2}\right)_{k\in\mathbb{N}}.$$

1. Show that T is a continuous linear operator and determine its norm.
2. Show that T is limit (in the operator sense) of finite rank operators. Is T compact?
3. Determine the set of eigenvalues and the spectrum of T.

12.3. Spectral decomposition of the Laplacian on an interval. The goal of this exercise is to show that there exists a Hilbert basis of eigenfunctions for the Laplace operator $-\frac{d^2}{dx^2}$ on $I = (0, \pi)$ with Dirichlet boundary conditions; namely there exists a Hilbert basis $\{e_n\} \subset L^2(I, \mathbb{R})$ and $\{\lambda_n\} \subset \mathbb{R}$ such that
$$\begin{cases} -e_n'' = \lambda_n e_n, \\ e_n(0) = e_n(\pi) = 0. \end{cases}$$

1. Let $f \in L^2(I, \mathbb{R})$ and consider the problem

$$\begin{cases} -u'' = f, \\ u(0) = u(\pi) = 0. \end{cases} \tag{1}$$

   Recall that in Exercise 6.3 we showed that $B = \{\alpha \sin(kx)\}_{k \geq 1}$, where $\alpha = \sqrt{2/\pi}$, is a Hilbert basis for $L^2([0, \pi], \mathbb{R})$. Write formally both u and f as a Fourier series on $(0, \pi)$ using the basis B and write a formal solution $u = u(f)$ of (1).

2. Give a sufficient condition on the Fourier coefficients of f to make sure that u is a classical solution of (1) (namely, u is at least C^2 and matches the boundary conditions)

3. Show that the formal solution u belongs to $L^2([0, \pi], \mathbb{R})$ and prove that the map $T\colon L^2([0, \pi], \mathbb{R}) \to L^2([0, \pi], \mathbb{R})$ defined by $T(f) = u(f)$ is continuous, self-adjoint and compact.

4. By the Spectral Theorem, we know that there is a Hilbert space consisting of eigenfunctions of T. Show that the same eigenfunctions are eigenfunctions of $-\frac{d^2}{dx^2}$.