The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

12.1. Closed answer questions.

1. Let (X, d) be a metric space and let $q \in X$. Assume you have a sequence $\{x_k\} \subset X$ such that *any* subsequence $\{x_{k_j}\}$ must possess a sub-subsequence $\{x_{j_{k_\ell}}\}$ such that $x_{j_{k_\ell}} \to q$ as $\ell \to \infty$. Is it true that $x_k \to q$ as $k \to \infty$?
2. Let $u \in \mathcal{S}(\mathbb{R})$, show that $\|\hat{u}\|^4_{L^4(\mathbb{R})} = 2\pi \|u * u\|^2_{L^2(\mathbb{R})}$.
3. Consider the set
$$X := \{u \in L^2(\mathbb{R}^d) : \hat{u}(\xi) \equiv 0 \text{ for almost every } |\xi| > 2\},$$
show that X is closed and not open in $L^2(\mathbb{R}^d)$. **Hint**: First show that $\mathcal{F}(X)$ is closed. Then...?
4. Consider the set
$$X := \{u \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) : \int_{\mathbb{R}} u = 0\},$$
show that X is dense in $L^2(\mathbb{R})$. What happens if we consider the exact same question in $[-\pi, \pi]$ instead of $\mathbb{R}$? **Hint**: again, try computing $\mathcal{F}(X)$ and use that $\mathcal{F}$ is an isometry.
5. Let H be an Hilbert space and $T \colon H \to H$ a linear operator of which we know the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ and the corresponding eigenfunctions $\{v_k\}_{k\in\mathbb{N}}$. Suppose that there exists an operator S such that $S \circ T = T \circ S = \mathrm{Id}$. Show that $\lambda_k \neq 0$ for every $k \in \mathbb{N}$ and determine the set of eigenvalues of S.

12.2. A simple compact operator on ℓ^2. Consider the map $T \colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by
$$T((x_k)_{k\in\mathbb{N}}) = \left(\frac{x_k}{k^2}\right)_{k\in\mathbb{N}}.$$

1. Show that T is a continuous linear operator and determine its norm.
2. Show that T is limit (in the operator sense) of finite rank operators. Is T compact?
3. Determine the set of eigenvalues and the spectrum of T.

12.3. Spectral decomposition of the Laplacian on an interval. The goal of this exercise is to show that there exists a Hilbert basis of eigenfunctions for the Laplace operator $-\frac{d^2}{dx^2}$ on $I = (0, \pi)$ with Dirichlet boundary conditions; namely there exists a Hilbert basis $\{e_n\} \subset L^2(I, \mathbb{R})$ and $\{\lambda_n\} \subset \mathbb{R}$ such that
$$\begin{cases} -e_n'' = \lambda_n e_n, \\ e_n(0) = e_n(\pi) = 0. \end{cases}$$

1. Let $f \in L^2(I, \mathbb{R})$ and consider the problem

$$\begin{cases} -u'' = f, \\ u(0) = u(\pi) = 0. \end{cases} \tag{1}$$

Recall that in Exercise 6.3 we showed that $B = \{\alpha \sin(kx)\}_{k\geq 1}$, where $\alpha = \sqrt{2/\pi}$, is a Hilbert basis for $L^2([0,\pi], \mathbb{R})$. Write formally both u and f as a Fourier series on $(0, \pi)$ using the basis B and write a formal solution $u = u(f)$ of (1).

2. Give a sufficient condition on the Fourier coefficients of f to make sure that u is a classical solution of (1) (namely, u is at least C^2 and matches the boundary conditions)

3. Show that the formal solution u belongs to $L^2([0,\pi], \mathbb{R})$ and prove that the map $T\colon L^2([0,\pi], \mathbb{R}) \to L^2([0,\pi], \mathbb{R})$ defined by $T(f) = u(f)$ is continuous, self-adjoint and compact.

4. By the Spectral Theorem, we know that there is a Hilbert space consisting of eigenfunctions of T. Show that the same eigenfunctions are eigenfunctions of $-\frac{d^2}{dx^2}$.

12. Solutions

Solution of 12.1:

1. Yes it is true. Assume by contradiction there is $\delta > 0$ such that $d(x_j, q) > \delta$ for infinitely many j's. Then we can extract a subsequence x_{j_k} that cannot have any sub-sequence converging to q, since it is always at least δ -distant from it, contradiction.

2. We have
$$\|\hat{u}\|_{L^4}^4 = \|\hat{u}^2\|_{L^2}^2 = 2\pi\|u * u\|_{L^2}^2.$$

3. $\mathcal{F}(X)$ contains all L^2 functions supported in the ball of radius 2, which is a closed subspace since if $v_j \in \mathcal{F}(X)$ and $v_j \to v$ in $L^2(\mathbb{R})$ then there is an appropriate subsequence $\{v_{j_k}\}$ which converges pointwise also almost everywhere to v, hence v must vanish outside B_2, that is $v \in \mathcal{F}(X)$. Since $X = \mathcal{F}^{-1}\mathcal{F}(X)$ and $\mathcal{F}^{-1}$ is an isometry we must have X closed.

   Since L^2 is connected if X was both open and closed it would be either empty or the full L^2, which is clearly not the case.

4. $\mathcal{F}(X)$ contains all the Schwartz functions vanishing at $\xi = 0$. This class is dense in L^2 so, $\mathcal{F}^{-1}$ being an isometry, necessarily X is dense in L^2.

   In the case of the interval X would not be dense at all being the kernel of the nontrivial continuous linear functional $u \mapsto \int_{-\pi}^{\pi} u$. The point is that the linear functional $u \mapsto \int_{\mathbb{R}} u$ is not bounded (nor well-defined) in $L^2(\mathbb{R})$.

5. Suppose by contradiction that $\lambda_{\bar{k}} = 0$ for some $\bar{k} \in \mathbb{N}$, this would mean that there exists a vector $v_{\bar{k}} \neq 0$ such that $Tv_{\bar{k}} = 0$. Applying S to this relation, we would find

$$v_{\bar{k}} = S \circ T(v_{\bar{k}}) = S(0) = 0,$$

which is a contradiction. Secondly, applying S to the relation

$$Tv_k = \lambda_k v_k$$

we find

$$\lambda_k S v_k = v_k$$

and using that $\lambda_k \neq 0$ for every $k \in \mathbb{N}$, we conclude that $\{1/\lambda_k\} \subseteq EV(S)$. On the other hand if there was an element $\mu \in EV(S)$ different from any $\{1/\lambda_k\}$ and with eigenfunction v_μ, then applying T to

$$Sv_\mu = \mu v_\mu$$

we would get that $1/\mu$ is an eigenvalue of T, which is a contradiction. Therefore, $EV(S) = \{1/\lambda_k\}$.

Solution of 12.2:

1. We readily check that

$$\|T(x)\|_{\ell^2}^2 = \sum_{k=1}^{\infty} \frac{x_k^2}{k^4} \leq \sum_{k=1}^{\infty} x_k^2 = \|x\|_{\ell_2}^2.$$

and that, defining $e_1 = (1, 0, \dots)$, $T(e_1) = e_1$. These two facts together imply $\|T\| = 1$.

2. For every $m \geq 1$, set $T_m \colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ as

$$T_m((x_k)_{k \in \mathbb{N}}) = \left(x_1, \frac{x_2}{2^2}, \dots, \frac{x_m}{m^2}, 0, 0, \dots \right).$$

It is clear that the range of T_m is contained in an m-dimensional subspace of $\ell^2(\mathbb{N})$, thus T_m is compact. We check that

$$\begin{aligned}
\|T_m - T\| &= \sup_{\|x\|_{\ell^2}=1} \|T_m(x) - T(x)\|_{\ell^2} \\
&= \sup_{\|x\|_{\ell^2}=1} \sqrt{\sum_{k \geq m+1} \frac{x_k^2}{k^4}} \\
&\leq \frac{1}{(m+1)^2} \sup_{\|x\|_{\ell^2}=1} \sqrt{\sum_{k \geq m+1} x_k^2} \\
&\leq \frac{1}{(m+1)^2} \to 0
\end{aligned}$$

as $m \to \infty$, i.e. $T_m \to T$ as operators. Since the class of compact operators is closed in the topology of the operator norm, T is compact.

3. An eigenvalue of T is an element $x \in \ell^2(\mathbb{N})$, $x \neq 0$ such that for some constant λ it holds
$$T(x) = \lambda x.$$
Passing to the coefficients, the eigenvalue equation reads
$$\frac{x_k}{k^2} = \lambda x_k \quad \forall k \geq 1. \tag{2}$$
If k is such that $x_k \neq 0$, this implies $\lambda = 1/k^2$; but then (2) forces $x_j = 0$ for every $j \neq k$. In particular, the eigenvectors of T are the elements $\{e_k\}$ of the standard basis and the corresponing eigenvalues are $1/k^2$. Lastly by Theorem 4.38, using compactness of T, we get that the spectrum is
$$\sigma(T) = \{0\} \cup \{1/k^2\}_{k\geq 1}.$$

Solution of 12.3:

1. We write formally
$$u(x) = \alpha \sum_{k\geq 1} u_k \sin(kx) \quad \text{and} \quad f(x) = \alpha \sum_{k\geq 1} f_k \sin(kx)$$
and we find
$$-u'' = \alpha \sum_{k\geq 1} k^2 u_k \sin(kx) = \alpha \sum_{k\geq 1} f_k \sin(kx) = f(x).$$
Thus, we define for every $f \in L^2(I, \mathbb{R})$ the function defined by the expression
$$T(f)(x) = u(f)(x) = \alpha \sum_{k\geq 1} \frac{f_k}{k^2} \sin(kx).$$

2. Observe that, if
$$\sum_{k\geq 1} k^2 \frac{|f_k|}{k^2} = \sum_{k\geq 1} |f_k| < \infty$$
by Theorem 2.26 we find that $u = T(f)$ is class C^2 and satisfy the boundary condition, therefore is a classical solution.

3. Observe that the series converges in L^2 as a consequence of Theorem 2.13 and the fact that
$$\alpha \sum_{k\geq 1} \left\| \frac{f_k}{k^2} \sin(kx) \right\|_{L^2}^2 = \sum_{k\geq 1} \left(\frac{f_k}{k^2} \right)^2 \leq \sum_{k\geq 1} f_k^2 < \infty.$$
The same calculation (and the "countable Pythagora's theorem" part of Theorem 2.13) shows also continuity of the map. Self-adjointness follows directly; if $u = T(f)$ and $v = T(g)$, then
$$\int_I T(f)g = \int_I uv'' = \int_I u''v = \int_I fT(g),$$

where we used twice integration by parts and the boundary conditions. Compactness can be shown directly but we present a different proof. By Corollary 1.56, $L^2(I, \mathbb{R})$ is isometric to $\ell^2_{\mathbb{R}}(\mathbb{N})$ via the isometry that maps any Hilbert basis B into $\{e_1, e_2, \dots\}$. In particular, the operator T is mapped into $\bar{T}$ defined by

$$\bar{T}(x) = (x_k/k^2)_{k\geq 1}.$$

By the previous exercise, $\bar{T}$ is compact, which implies the same for T.

4. By differentiating twice the eigenvalue equation $\mu_k u_k = T u_k$ we find

$$\mu_k u_k'' = (T u_k)'' = -u_k$$

thus, u_k is also an eigenfunction of $-\frac{d^2}{dx^2}$ with eigenvalue $1/\mu_k$.