The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

1.1. Inner product spaces.

1. Let $V := \mathrm{M}_{n\times n}(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries and define the *Fobenius product* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ as
$$\langle A, B\rangle := \mathrm{Tr}(AB^\dagger) = \sum_{i,j=1}^{n} a_{ij}\bar{b}_{ij}$$
where Tr denotes the trace and $B^\dagger$ is the Hermitian transpose of B, obtained by transposition and complex conjugation of the entries: $B^\dagger = \overline{B^T}$. Show that $(V, \langle \cdot, \cdot \rangle)$ is an inner-product space. **Hint:** first observe that $\mathrm{Tr}(A) = \overline{\mathrm{Tr}(A^\dagger)}$.
2. Consider n inner-product spaces $(V_1, \langle \cdot, \cdot \rangle_1), \ldots, (V_n, \langle \cdot, \cdot \rangle_n)$. Is $(V, \langle \cdot, \cdot \rangle)$, where $V = V_1 \times \cdots \times V_n$ and
$$\langle (v_1, \ldots, v_n), (w_1, \ldots, w_n)\rangle := \sum_{i=1}^{n} \langle v_i, w_i\rangle_i,$$
an inner product space?
3. Let $W := M_{n\times n}(L^2(\mathbb{R}, \mathbb{C}))$ be the space of $n \times n$ matrices whose entries are square integrable functions from $\mathbb{R}$ to $\mathbb{C}$. Which product would make W an inner product space? **Hint:** observe that W is a "composition" of two inner product spaces.

1.2. Continuity of operations. An inner product space $(V, \langle \cdot, \cdot \rangle)$ is also a metric space under the norm $|\cdot| := \sqrt{\langle \cdot, \cdot \rangle}$, hence it has a natural topology. Prove that $\langle \cdot, \cdot \rangle$ and the vector space operations $(\cdot, +)$ are continuous from $V \times V$ (resp. $V \times \mathbb{C}$, $V \times V$) endowed with the natural product topology, to $\mathbb{C}$ (resp. V, V). Recall that a natural topology in $V \times V$ is the one induced by $|\cdot|$, i.e. the one induced by the norm
$$|(v_1, v_2)|_{V\times V} := |v_1| + |v_2|.$$
Similarly, the norm (thus the metric and the topology) on $V \times \mathbb{C}$ is given by
$$|(v, \alpha)|_{V\times\mathbb{C}} := |v| + |\alpha|.$$
Hint: is there a clever way to write $\langle \cdot, \cdot \rangle$, in order to prove continuity?

1.3. Topology of normed spaces. Determine whether the following sets X are well-defined, open, close, subspaces and convex.

1. In the normed space $(C([0,1]), \|\cdot\|_{L^\infty})$, the subset X of nowhere vanishing functions.
2. In the normed space $(C([0,1]), \|\cdot\|_{L^2})$, the subset X of nowhere vanishing functions.
3. (BONUS) In the normed space $(L^2(0,1), \|\cdot\|_{L^2})$, the subset $X = \{f : \int_0^1 f = 1\}$.
   - ☐ Not well defined.
   - ☐ Well defined, open and convex.
   - ☐ Well defined, closed, convex but not a linear subspace.
   - ☐ Well defined, closed and linear subspace.
4. In the normed space $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$, the subset $\{f : f(x) = f(-x) \text{ for a.e. } x \in \mathbb{R}\}$. **Hint:** It's useful to recall that if $u_k \to u$ in L^2 then, up to picking a subsequence, there is a null measure set N such that $u_k(x) \to u(x)$ for all $x \notin N$.
5. $(*)$ In the normed space $(L^2(0,1), \|\cdot\|_{L^2})$, the subset $X = \{f : f \geq 0 \text{ and } \int_0^1 \frac{2f}{1+f} \geq 1\}$. **Hint:** observe that the map $s \mapsto 2s/(1+s)$ is concave for $s \geq 0$.

1.4. Quantitative Cauchy Schwarz. Let H be a real inner product space, prove the identity

$$|x||y| - x \cdot y = \frac{|x||y|}{2}\left|\frac{x}{|x|} - \frac{y}{|y|}\right|^2 \geq 0 \text{ for all } x, y, \in H.$$

Characterize the set $C \subset H \times H$ of pair of vectors that saturate the Cauchy-Schwarz inequality, i.e. $x \cdot y = |x||y|$. Plot C in the case $H = \mathbb{R}$.

$(*)$ If x, y are ϵ-close to saturate the Cauchy Schwarz inequality, that is

$$(1-\epsilon)|x||y| \leq x \cdot y,$$

then how close are x, y to the set C? Bound from above the number

$$\inf_{(x',y') \in C} |x - x'|^2 + |y - y'|^2 =: \operatorname{dist}^2((x,y), C).$$

1. Solutions

Solution of 1.1:

1. We need to check that $\langle \cdot, \cdot \rangle$ satisfies the three axioms of inner product space: conjugate symmetry, linearity in the first argument and positive definiteness. First observe that the property $\text{Tr}(A) = \overline{\text{Tr}(A^\dagger)}$ follows directly from the fact that the trace is invariant by transposition. Remark also that Hermitian transposition is an involution, i.e. $(B^\dagger)^\dagger = B$. Then

$$\langle A, B\rangle = \text{Tr}(AB^\dagger) = \overline{\text{Tr}((AB^\dagger)^\dagger)} = \overline{\text{Tr}((B^\dagger)^\dagger A^\dagger)} = \overline{\text{Tr}(BA^\dagger)} = \overline{\langle B, A\rangle}$$

and hence conjugate symmetry holds. Linearity in the first argument follows trivially by linearity of the trace and, if $A \neq 0$,

$$\text{Tr}(AA^\dagger) = \sum_{i,j=1}^{n} a_{ij}\overline{a_{ij}} = \sum_{i,j=1}^{n} |a_{ij}|^2 > 0.$$

which proves positive definiteness.

2. Yes, it is. Denote $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. We readily check that

$$\langle v, w\rangle = \sum_i \langle v_i, w_i\rangle_i = \sum_i \overline{\langle w_i, v_i\rangle_i} = \overline{\langle w, v\rangle}$$

and

$$\langle \alpha v + \beta u, w\rangle = \sum_i \langle \alpha v_i + \beta u_i, w_i\rangle_i = \sum_i \alpha\langle v_i, w_i\rangle_i + \beta\langle u_i, w_i\rangle_i = \alpha\langle v, w\rangle + \beta\langle u, w\rangle$$

which prove the first two axioms. Positive definiteness follows from the fact that $v \neq 0 \iff v_k \neq 0$ for some $k \in \{1, \ldots, n\}$. Thus

$$\langle v, v\rangle = \sum_{i=1}^{n} \langle v_i, v_i\rangle_i \geq \langle v_k, v_k\rangle_k > 0.$$

3. Let $F = (f_{ij})_{ij}$ and $G = (g_{ij})_{ij}$ be two matrices of square integrable functions. Define

$$\langle F, G\rangle = \sum_{i,j=1}^{n} \int_{\mathbb{R}} f_{ij}(x)\overline{g_{ij}(x)}dx.$$

This product is a natural choice since it's the "composition" of the Frobenius product (defined above) and the L^2 inner product. It's direct to check that all the axioms of inner product space hold.

Remark: this "composition trick" was implicitly used also in part 2. Indeed the inner product there is a composition of the ones of the respective V_i and the one in $\mathbb{R}^n$, given by $a \cdot b = \sum_i a_i b_i$.

Solution of 1.2: The sum is continuous (Lipschitz, even) by the triangular inequality

$$|(v_1 + v_2) - (u_1 + u_2)| \leq |v_1 - u_1| + |v_2 - u_2| = |(v_1, v_2) - (u_1, u_2)|_{V \times V}.$$

So if (v_1, v_2) and (u_1, u_2) are ϵ-close in $V \times V$ (i.e., the right hand side is smaller than ϵ) then $v_1 + v_2$ is ϵ-close to $u_1 + u_2$ in V. Multiplication by a scalar is also continuous by homogeneity and the triangular inequality

$$|\alpha v - \alpha' v'| \leq |\alpha - \alpha'||v| + |\alpha'||v - v'| \leq \max\{|\alpha'|, |v|\}|(v, \alpha) - (v', \alpha')|_{V \times \mathbb{C}}$$

which proves the continuity of the multiplication by a scalar.

By the polarisation identities the scalar product then needs to be continuous, since it is a composition of continuous functions.

Solution of 1.3:

1. It is well defined and open, but neither close nor convex (and hence not a linear subspace as well). We show openness: if $u \in X$ then $\delta := \min_{[0,1]} |u|$ is strictly positive, so for any other $v \in C([0,1])$, with $\|u - v\|_{L^\infty(0,1)} < \delta/100$, we find for all $x \in [0,1]$ that

   $$|v(x)| \geq |u(x)| - \|u - v\|_{L^\infty(0,1)} \geq \delta - \frac{\delta}{100} > \delta/2,$$

   so $v \in X$. It is not closed since the functions $f_k = 2^{-k}$ belong to X but their limit does not, and X is not convex since $u \in X, -u \in X, u + (-u) = 0 \notin X$.

2. It is not open nor closed and not convex.

   It is not closed nor convex by the same examples as above.

   It is not open because if $u \in X, u > 0$ and $\epsilon > 0$ small then $u_\epsilon(x) := \min\{x/\epsilon, u(x)\}$ does not lie in X, but it is as close as we want to u since

   $$\begin{aligned}\limsup_\epsilon \|u - u_\epsilon\|^2_{L^2(0,1)} &= \limsup_\epsilon \int_{\{x : u(x) > x/\epsilon\}} (x/\epsilon)^2 \, dx \\ &\leq \limsup_\epsilon \int_{\{x : u(x) > x/\epsilon\}} u(x)^2 \, dx = 0,\end{aligned}$$

   by dominated convergence, since

   $$\sup_{0 < \epsilon < 1/2} \mathbf{1}_{\{x : u(x) > x/\epsilon\}} u(x)^2 \leq u(x)^2 \in L^1(0,1)$$

   and for each fixed z we have $\lim_\epsilon \mathbf{1}_{\{x : u(x) > x/\epsilon\}}(z) = 0$.

3. X is well-defined, closed and convex, but not a linear space as $0 \notin X$. It's well-defined since $L^2(0,1) \subset L^1(0,1)$. It's closed because if $u_k \in X$ and $u_k \to u$ in L^2 then

   $$\Big|\int u_k - \int u\Big| \leq \int |u_k - u| = \|u - u_k\|_{L^1} \leq \|u_k - u\|_{L^2} \to 0.$$

This means that $\int u = \lim_k \int u_k = \lim_k 1 = 1$. This proves that $u \in X$, hence X contains its accumulation points, hence it is closed. Since $X \neq L^2$, X is closed and L^2 is connected, then X cannot be open; alternatively you can show that the complement of X is not closed but taking, e.g., $f_k = 1 + 2^{-k}$.

Convexity is immediately checked by linearity of the integral

$$\begin{aligned} &u \in X, v \in X, t \in [0,1] \Rightarrow \\ \int (tu + (1-t)v) = t \int u + (1-t) \int v &= t + 1 - t = 1 \\ &\Rightarrow tu + (1-t)v \in X. \end{aligned}$$

4. It is well-defined, closed and a linear subspace.

   Well-defined because if u and u' agree almost everywhere then also $u(-\cdot)$ and $u'(-\cdot)$ do. This is because if $A \subset \mathbb{R}$ has full measure then also $-A$ has full measure and so does $A \cap (-A)$.

   Closed since arguing by sub-sequences we find $N \subset \mathbb{R}$ with $|N| = 0$ such that

   $$u_k(x) \to u(x) \text{ and } u_k(-x) \to u(-x) \text{ for all } x \in \mathbb{R} \setminus N.$$

   Thus $0 \equiv u_k(x) - u_k(-x) \to u(x) - u(-x) = 0$.

   The fact that $0 \in X$ and is closed by linear combinations is immediate to check, let us refresh the full argument which you probably have seen in Analysis III: if

   $$u(x) = u(-x) \text{ for all } x \in \mathbb{R} \setminus N_u \text{ and } v(x) = v(-x) \text{ for all } x \in \mathbb{R} \setminus N_v,$$

   with $|N_u| = |N_v| = 0$ then for all $x \in \mathbb{R} \setminus (N_u \cup N_v)$ we have

   $$\alpha u(x) + \beta v(x) = \alpha u(-x) + \beta v(-x).$$

5. X is well defined, closed and convex.

   Well defined because for all $u \in L^2, u \geq 0$ it holds

   $$\int_0^1 \frac{|2u(t)|}{|1+u(t)|}\,dt \leq \int_0^1 2\,dt = 2$$

   as for any nonnegative number $|2u/(1+u)| \leq 2$.

   Convex because the function $\psi \colon s \mapsto \frac{2s}{1+s}$ is concave for $s \in [0,\infty)$. So if $u, v \in X$ and $t \in [0,1]$ we have for almost every x

   $$\begin{aligned} \frac{tu(x) + (1-t)v(x)}{1 + tu(x) + (1-t)v(x)} &= \psi(tu(x) + (1-t)v(x)) \\ &\geq t\psi(u(x)) + (1-t)\psi(v(x)) = t\frac{u(x)}{1+u(x)} + (1-t)\frac{v(x)}{1+v(x)}, \end{aligned}$$

integrating both sides of this inequality we find

$$\int_0^1 2\frac{tu(x)+(1-t)v(x)}{1+tu(x)+(1-t)v(x)} \geq t\int_0^1 \frac{2u(x)}{1+u(x)}+(1-t)\int_0^1 \frac{2v(x)}{1+v(x)} \geq t\cdot 1+(1-t)\cdot 1 = 1,$$

which means that $tu+(1-t)v \in X$.

To check closeness we pick a sequence $u_k \in X$ with $u_k \to u$ in L^2. We want to show that also $u \in X$. We take a subsequence (which we don't re-label) and gain that also $u_k(x) \to u(x)$ for all $x \in (0,1) \setminus N$ with $|N| = 0$. Since u_k were nonnegative we find that also $u \geq 0$ a.e. It remains to show that $\int \frac{u}{1+u} \geq 1$. To do so we invoke the dominated convergence theorem, i.e. since the integrands are bounded by a common function

$$\frac{2|u_k|}{|1+u_k|} \leq 2 \text{ uniformly in } k,$$

we can exchange pointwise limit and integral and find

$$\int_0^1 \frac{2u}{1+u} = \lim_k \int_0^1 \frac{2u_k}{1+u_k} \geq 1.$$

Solution of 1.4: The identity is equivalent to prove that

$$2|x|^2|y|^2 - 2(x\cdot y)|x||y| \stackrel{?}{=} |x|y| - y|x||^2$$

which is straightforward to check expanding the square at the right hand side.

For the first part, using the identity one finds

$$C = \{(\alpha\xi, \beta\xi) : \alpha \geq 0, \beta \geq 0, |\xi| = 1\}.$$

That is, x, y need to be parallel and equi oriented in order to saturate C.S.

For the second part, we plug-in the inequality in the previous identity and get

$$\frac{|x||y|}{2}\left|\frac{x}{|x|} - \frac{y}{|y|}\right|^2 \leq \epsilon|x||y|,$$

which gives

$$\left|\frac{x}{|x|} - \frac{y}{|y|}\right| \leq \sqrt{2\epsilon}.$$

Since $x/|x|$ and $y/|y|$ are the directions of the vectors x, y this is telling us that if we ϵ-saturate the C.S. inequality, then we are $O(\sqrt{\epsilon})$ far away from the equality case (i.e., x and y being parallel).

This would be already a somewhat satisfying answer, but we want to estimate

$$\mathrm{dist}((x,y),C)^2 := \inf_{(x',y')\in C} |x-x'|^2 + |y-y'|^2.$$

In order to do so, we plug $x' \leftarrow \frac{|x|}{|y|}y$ and $y' \leftarrow y$ (and the symmetric assignment) to find

$$\operatorname{dist}((x,y),C)^2 \le \min\{|x|^2,|y|^2\}\left|\frac{x}{|x|}-\frac{y}{|y|}\right|^2 \le 2\epsilon\min\{|x|^2,|y|^2\} \le 2\epsilon|x||y|,$$

where in the last inequality we used that for any pair of nonnegative numbers $\min\{a,b\} \le \sqrt{ab}$.

In other words, we proved the *quantitative* Cauchy-Schwarz inequality:

$$0 \le \frac{1}{2}\operatorname{dist}((x,y),C)^2 \le |x||y| - x\cdot y \quad \text{for all } x,y,\in H.$$