The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

2.1. Scalar products and Hilbert spaces. Prove or disprove whether the following pairs (vector space, bilinear form) are Hilbert spaces. Additionally, write down what the squared norm of a vector is in each case.

1. $V := L^2(\mathbb{R};\mathbb{C})$ and $\langle u,v\rangle := \int_{\mathbb{R}} u(t)\bar{v}(t)\frac{dt}{1+t^2}$
2. $V :=$ {real polynomials of degree at most N} and[1] $\langle p,q\rangle := p(\frac{d}{dx})|_{x=0}q$. **Hint:** observe that $\left(\frac{d}{dx}\right)^j_{x=0} x^k = \delta^{kj}\, k!$
3. $V := L^1(0,1)$ and $\langle u,v\rangle := \int_0^1 u(x)v(x)\,dx$.
4. $V := \mathbb{Q}^d$ and $\langle x,y\rangle := \sum_{k=1}^d x_k y_k$.

2.2. Inner product from the norm. Let

$$V := \left\{u \in C^2((0,1)) \cap C([0,1]) : u(0) = 0\right\}$$

Determine which of the following maps $\|\cdot\|\colon V \to \mathbb{R}$ defines a norm over V (no need to check completeness)[2].

A. $\|u\|_A = \left(\int_0^1 |u''(x)|^2 dx\right)^{1/2}$

B. $\|u\|_B = \left(\int_0^1 |u'(x)|^2 dx\right)^{1/2}$

C. $\|u\|_C = \left(\int_0^1 |u'(x)|^3 dx\right)^{1/3}$

D. $\|u\|_D = \left(\int_0^1 \int_0^1 \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy\right)^{1/2}$

(BONUS) Which of the above expression defines a norm on V that arises from an inner product space? You can choose multiple answers. **Hint:** Recall the parallelogram law.

☐ A ☐ B ☐ C ☐ D

[1] If $p(X)$ is a polynomial, then $p(\frac{d}{dx})|_{x=0}$ is the differential operator obtained replacing X by $\frac{d}{dx}$ and then evaluating at $x=0$. Example: if $p(X) = X^2+3$ then $p(\frac{d}{dx})|_{x=0}q = q''(0) + 3q(0)$.

[2] Recall Minkowski inequality: for $p \in (1,+\infty)$ and $f,g \in L^p(X,\mu)$, then $(\int_X |f+g|^p d\mu)^{1/p} \le (\int_X |f|^p d\mu)^{1/p} + (\int_X |g|^p d\mu)^{1/p}$.

2.3. Legendre Polynomials I. Consider the Hilbert space $H := L^2((-1,1), dx)$. Apply the Gram-Schmidt algorithm to the ordered set $\{1, x, x^2\} \subset H$, and find three orthonormal polynomials $e_0(x), e_1(x), e_2(x)$.

2.4. Legendre Polynomials II. Consider in the Hilbert space $H := L^2((-1,1), dx)$ the polynomials

$$P_0 := 1, \quad P_k(x) := D^k((x^2-1)^k) \text{ for } k \geq 1,$$

where $D := d/dx$. The first goal is to prove that $\{P_j\}_{j\geq 0}$ is an orthonormal system. You can follow this outline

1. Show that each P_k has degree k and show that $D^k P_k(x) = (2k)!$.
2. Show that for $0 \leq k' < k$ the function $D^{k'}((x^2-1)^k)$ vanishes at ± 1 (**Hint:** use the Leibniz formula: $D^k(f \cdot g) = \sum_{j=0}^k \binom{k}{j} D^j f \cdot D^{k-j} g$);
3. Use the previous point and multiple integration by parts to show that if $0 \leq k < k'$ then
$$\int_{-1}^1 P_k(x) P_{k'}(x)\, dx = 0.$$

In order to have a orthonormal basis we need to compute $\|P_k\|_{L^2(-1,1)}$. You can follow this outline

4. Given for granted[3] that $B(k+1, k+1) := \int_0^1 s^k (1-s)^k\, ds = \frac{(k!)^2}{(2k+1)!}$, show that
$$\int_{-1}^1 (x^2-1)^k\, dx = (-1)^k \frac{2^{2k+1}(k!)^2}{(2k+1)!}$$
5. Using multiple times integration by parts and the previous points show that
$$\|P_k\|^2_{L^2(-1,1)} = \int_{-1}^1 P_k(x)^2\, dx = (-1)^k \int_{-1}^1 (x^2-1)^k D^k P_k(x)\, dx = \frac{2^{2k+1}(k!)^2}{2k+1}$$
6. $(*)$ Prove that $B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$ for all $n, m \geq 1$. **Hint**: you might want to prove it first for $B(0,m)$ and then find a formula (integrating by parts) that relates $B(n,m)$ with $B(n-1, m+1)$ and proceed inductively.
7. Finally, (double) check that indeed
$$e_0(x) = \frac{P_0(x)}{\|P_0\|_{L^2(-1,1)}}, \quad e_1(x) = \frac{P_1(x)}{\|P_1\|_{L^2(-1,1)}} \text{ and } e_2(x) = \frac{P_2(x)}{\|P_2\|_{L^2(-1,1)}},$$
where e_0, e_1, e_2 are the polynomials of exercise 2.3. Is that a coincidence that they are the same?

[3]This is a value of the so-called Euler's Beta function $B(x,y) := \int_{-1}^1 t^{x-1}(1-t)^{y-1}\, dt = B(y,x)$.