The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

2.1. Scalar products and Hilbert spaces. Prove or disprove whether the following pairs (vector space, bilinear form) are Hilbert spaces. Additionally, write down what the squared norm of a vector is in each case.

1. $V := L^2(\mathbb{R};\mathbb{C})$ and $\langle u, v\rangle := \int_{\mathbb{R}} u(t)\bar{v}(t)\frac{dt}{1+t^2}$
2. $V := \{\text{real polynomials of degree at most } N\}$ and[1] $\langle p, q\rangle := p(\frac{d}{dx})|_{x=0}q$. **Hint:** observe that $\left(\frac{d}{dx}\right)^j_{x=0} x^k = \delta^{kj}\, k!$
3. $V := L^1(0,1)$ and $\langle u, v\rangle := \int_0^1 u(x)v(x)\,dx$.
4. $V := \mathbb{Q}^d$ and $\langle x, y\rangle := \sum_{k=1}^d x_k y_k$.

2.2. Inner product from the norm. Let

$$V := \left\{u \in C^2((0,1)) \cap C([0,1]) : u(0) = 0\right\}$$

Determine which of the following maps $\|\cdot\| \colon V \to \mathbb{R}$ defines a norm over V (no need to check completeness)[2].

A. $\|u\|_A = \left(\int_0^1 |u''(x)|^2 dx\right)^{1/2}$

B. $\|u\|_B = \left(\int_0^1 |u'(x)|^2 dx\right)^{1/2}$

C. $\|u\|_C = \left(\int_0^1 |u'(x)|^3 dx\right)^{1/3}$

D. $\|u\|_D = \left(\int_0^1 \int_0^1 \frac{|u(x)-u(y)|^2}{|x-y|^2} dx dy\right)^{1/2}$

(BONUS) Which of the above expression defines a norm on V that arises from an inner product space? You can choose multiple answers. **Hint:** Recall the parallelogram law.

☐ A ☐ B ☐ C ☐ D

[1]If $p(X)$ is a polynomial, then $p(\frac{d}{dx})|_{x=0}$ is the differential operator obtained replacing X by $\frac{d}{dx}$ and then evaluating at $x = 0$. Example: if $p(X) = X^2 + 3$ then $p(\frac{d}{dx})|_{x=0}q = q''(0) + 3q(0)$.

[2]Recall Minkowski inequality: for $p \in (1, +\infty)$ and $f, g \in L^p(X, \mu)$, then $(\int_X |f + g|^p d\mu)^{1/p} \leq (\int_X |f|^p d\mu)^{1/p} + (\int_X |g|^p d\mu)^{1/p}$.

2.3. Legendre Polynomials I. Consider the Hilbert space $H := L^2((-1,1), dx)$. Apply the Gram-Schmidt algorithm to the ordered set $\{1, x, x^2\} \subset H$, and find three orthonormal polynomials $e_0(x), e_1(x), e_2(x)$.

2.4. Legendre Polynomials II. Consider in the Hilbert space $H := L^2((-1,1), dx)$ the polynomials

$$P_0 := 1, \quad P_k(x) := D^k((x^2-1)^k) \text{ for } k \geq 1,$$

where $D := d/dx$. The first goal is to prove that $\{P_j\}_{j\geq 0}$ is an orthonormal system. You can follow this outline

1. Show that each P_k has degree k and show that $D^k P_k(x) = (2k)!$.
2. Show that for $0 \leq k' < k$ the function $D^{k'}((x^2-1)^k)$ vanishes at ± 1 (**Hint:** use the Leibniz formula: $D^k(f \cdot g) = \sum_{j=0}^k \binom{k}{j} D^j f \cdot D^{k-j} g$);
3. Use the previous point and multiple integration by parts to show that if $0 \leq k < k'$ then
$$\int_{-1}^1 P_k(x) P_{k'}(x)\, dx = 0.$$

In order to have a orthonormal basis we need to compute $\|P_k\|_{L^2(-1,1)}$. You can follow this outline

4. Given for granted[3] that $B(k+1, k+1) := \int_0^1 s^k(1-s)^k\, ds = \frac{(k!)^2}{(2k+1)!}$, show that
$$\int_{-1}^1 (x^2-1)^k\, dx = (-1)^k \frac{2^{2k+1}(k!)^2}{(2k+1)!}$$
5. Using multiple times integration by parts and the previous points show that
$$\|P_k\|^2_{L^2(-1,1)} = \int_{-1}^1 P_k(x)^2\, dx = (-1)^k \int_{-1}^1 (x^2-1)^k D^k P_k(x)\, dx = \frac{2^{2k+1}(k!)^2}{2k+1}$$
6. $(*)$ Prove that $B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$ for all $n, m \geq 1$. **Hint**: you might want to prove it first for $B(0,m)$ and then find a formula (integrating by parts) that relates $B(n,m)$ with $B(n-1, m+1)$ and proceed inductively.
7. Finally, (double) check that indeed
$$e_0(x) = \frac{P_0(x)}{\|P_0\|_{L^2(-1,1)}}, \quad e_1(x) = \frac{P_1(x)}{\|P_1\|_{L^2(-1,1)}} \text{ and } e_2(x) = \frac{P_2(x)}{\|P_2\|_{L^2(-1,1)}},$$
where e_0, e_1, e_2 are the polynomials of exercise 2.3. Is that a coincidence that they are the same?

[3] This is a value of the so-called Euler's Beta function $B(x,y) := \int_{-1}^1 t^{x-1}(1-t)^{y-1}\, dt = B(y,x)$.

2. Solutions

Solution of 2.1:

1. This is a legit inner product space and the proof of it is a simple consequence of the properties of inegrals and complex conjugation. It is not complete: the completion is the space $L^2(\mathbb{R},\mu)$ with respect to the measure $\mu = \frac{dt}{1+t^2}$, essentially by definition. To see this concretely take $u_\epsilon(t) := e^{-\epsilon t^2} \in V$ and we have that $u_\epsilon \to 1$ in with respect to the norm induced by the inner product
$$\|f\|^2 = \int_{\mathbb{R}} \frac{|f(t)|^2}{1+t^2}\, dt.$$
Indeed
$$\int_{\mathbb{R}} \frac{|u_\epsilon(t)-1|^2}{1+t^2} dt \to 0$$
because of dominated convergence theorem, since $u_\epsilon \leq 1 \in L^2(\mathbb{R},\mu)$. But then V cannot be complete because $\{u_\epsilon\}$ is Cauchy (being convergent), but $1 \notin V$.

2. We first check that this is an inner product space, then completeness follows from finite dimensionality. Linearity follows from the linearity of differentiation, so we just have to check that the given bilinear form is positive definite and symmetric. For $p, q \in V$ we write
$$p(X) = \sum_{j=0}^{N} p_j X^j, \qquad q(X) = \sum_{j=0}^{N} q_j X^j,$$
and compute
$$\langle p, q\rangle = \sum_{j=0}^{N} p_j q^{(j)}(0) = \sum_{j=0}^{N} p_j q_j\, j!$$
which is clearly symmetric in $p \leftrightarrow q$. We used that $\left(\frac{d}{dx}\right)^j_{x=0} x^k = \delta^{kj}\, k!$. We remark that in the basis $[1, X, \dots, X^N]$ the scalar product is given by the matrix
$$\mathrm{diag}[0!, 1!, 2!, \dots, N!],$$
which is positive definite having positive eigenvalues (recall $0! = 1$). The norm squared is
$$\|p\|^2 = \sum_{j=0}^{N} |p_j|^2\, j!.$$

3. No, the given bilinear form is not even well defined as the integral might diverge. For instance, if $u(x) = x^{-1/2}$ then $\langle u, u\rangle = +\infty$.

4. It is not even an $\mathbb{R}$-vector space, as the multiplication of a rational number by a real number does not give, in general, a rational number.

   If we think $\mathbb{Q}^d$ as a vector space over $\mathbb{Q}$ the scalar product is well defined, so we have a norm, but the resulting space is not complete, for the same reason for which $\mathbb{Q}$ is not complete as a subset of $\mathbb{R}$.

Solution of 2.2: The only one that is not a norm is A, since positivity fails. Indeed, the function $f(x) = x$ belongs to V and $\|f\|_A = 0$, but $f \neq 0$. The norm axioms for the norms B, C and D are readily checked. For any $\bullet \in \{\mathrm{B}, \mathrm{C}, \mathrm{D}\}$, $\|\cdot\|_\bullet \geq 0$ and for any scalar α

$$\|\alpha u\|_\bullet = |\alpha| \|u\|_\bullet .$$

Triangle inequality follows from Minkowski inequality in L^p spaces. Finally, by nonnegativity of the integral and continuity of the function we infer that

- If $\|u\|_B = 0$ or $\|u\|_C = 0$, then $u' \equiv 0$ and since $u(0) = 0$ we deduce $u \equiv 0$.
- If $\|u\|_D = 0$, then $u(x) - u(y) = 0$ for every $x, y \in [0, 1]$, thus u is constant. Again, using that $u(0) = 0$ we infer that $u \equiv 0$.

The norms arising from an inner product space are B and D. In fact the parallelogram law holds: for B we have

$$\begin{aligned}
\|u+v\|_B^2 + \|u-v\|_B^2 &= \int_0^1 |u'+v'|^2 + \int_0^1 |u'-v'|^2 \\
&= \int_0^1 |u'|^2 + \int_0^1 |v'|^2 + 2\int_0^1 u'v'd \\
&+ \int_0^1 |u'|^2 + \int_0^1 |v'|^2 - 2\int_0^1 u'v'd \\
&= 2\int_0^1 |u'|^2 + 2\int_0^1 |v'|^2 = 2\|u\|_B^2 + 2\|v\|_B^2
\end{aligned}$$

and a completely analogous calculation holds for D. $\|\cdot\|_C$ does not satisfy the parallelogram law, indeed take for instance $u(x) = x$ and $v(x) = \frac{x^2}{2}$. Then

$$\begin{aligned}
\|u+v\|_C^2 + \|u-v\|_C^2 &= \left(\int_0^1 |1+x|^3 dx\right)^{2/3} + \left(\int_0^1 |1-x|^3 dx\right)^{2/3} = (15/4)^{2/3} + (1/4)^{2/3} \\
2\|u\|_C^2 + 2\|v\|_C^2 &= 2\left(\int_0^1 dx\right)^{2/3} + 2\left(\int_0^1 |x|^3 dx\right)^{2/3} = 2 + 2(1/4)^{2/3}
\end{aligned}$$

and the two values differ.

EDIT: It was correctly pointed out by some students that in the space V the above norms are not necessarily well defined, as the example of $\sqrt{x}$ (or similar) shows. This is amended by requiring the derivatives of u to be continuous in the whole closed set $[0, 1]$. Since this was not specified, giving as answer B and D, B, D or none will grant the point.

Solution of 2.3: The polynomials 1 and x are already orthogonal in H for parity reasons, so normalizing them we find

$$e_0(x) = 1/\sqrt{2}, e_1(x) = \sqrt{\frac{3}{2}}x.$$

Also x^2 and x are orthogonal and

$$\langle x^2, e_0(x)\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} x^2\,dx = \sqrt{2}/3, \quad \langle x^2, e_1(x)\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1} x^3\,dx = 0.$$

and so

$$P_{\mathrm{span}\{e_0,e_1\}^\perp}(x^2) = x^2 - \langle x^2, e_0\rangle e_0 - \langle x^2, e_1\rangle e_1 = x^2 - 1/3.$$

Now we compute the norm

$$\int_{-1}^{1} (x^2 - 1/3)^2\,dx = 8/45,$$

and we find $e_2(x) = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)$.

Solution of 2.4: We start with proving that it is an orthonormal system.

1. It is known that taking the derivative of a polynomial of degree k yields a polynomial of degree $k-1$. Since $(x^2-1)^k$ has degree $2k$, taking k derivatives gives a polynomial of degree $2k - k = k$.

   Using the binomial formula we have

$$D^k P_k = D^{2k}((x^2-1)^k) = D^{2k}x^{2k} + \underbrace{D^{2k}(\text{poly of degree } 2k-1)}_{=0} = (2k)!.$$

2. Let us do the computation for $k' = k-1$. We compute the derivative with Leibniz' formula

$$\begin{aligned} D^{k-1}((x^2-1)^k) &= D^{k-1}((x+1)^k(x-1)^k) \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} D^j((x-1)^k) D^{k-1-j}((x+1)^k) \\ &= \sum_{j=0}^{k=1} c_{k,j}(x-1)^{k-j}(x+1)^{j+1}, \end{aligned}$$

   for some combinatorial constants $c_{n,k} \in \mathbb{N}$. Evaluating at $x = 1$ (or at $x = -1$) we see that all the terms in the sum vanish as $k - j \geq 1$ (and $j + 1 \geq 1$).

3. We integrate by parts $k+1$ times, using that the boundary terms vanish thanks to the previous point

$$\begin{aligned} \int_{-1}^{1} P_k\,P_{k'} &= \int_{-1}^{1} D^k((x^2-1)^k) D^{k'}((x^2-1)^{k'}) \\ &= \underbrace{\Big[D^k((x^2-1)^k) D^{k'-1}((x^2-1)^{k'})\Big]_{-1}^{+1}}_{=0} - \int_{-1}^{1} D^{k+1}((x^2-1)^k) D^{k'}((x^2-1)^{k'}) \\ &= \cdots \\ &= (-1)^{k+1}\int_{-1}^{1} \underbrace{D^{2k+1}((x^2-1)^k)}_{=0} D^{k'-k-1}((x^2-1)^{k'}) \end{aligned}$$

4-5 We compute the norm doing the same computation we just did, but integrating by parts k times:

$$\begin{aligned}\int_{-1}^{1} P_k^2 &= (-1)^k \int_{-1}^{1} D^{2k}((x^2-1)^k)(x^2-1)^k\,dx \\ &= (-1)^k(2k)! \int_{-1}^{1} (x^2-1)^k\,dx \qquad [x+1=2t] \\ &= (-1)^k(2k)!2^{2k+1} \int_0^1 t^k(t-1)^k\,dt \\ &= (2k)!\ 2^{2k+1} \int_0^1 t^k(1-t)^k\,dt = (2k)!2^{2k+1}B(k+1,k+1).\end{aligned}$$

6. We need to prove the formula for $B(k+1,k+1)$, and it turns out to be easier to prove it directly for $B(n,m)$. We set for simplicity $B_{n,m} := B(n+1,m+1)$ and integrating by parts once again

$$\begin{aligned}B_{n,m} &:= \int_0^1 t^n(1-t)^m = \frac{1}{n+1}\int_0^1 (t^{n+1})'(1-t)^m \\ &= -\frac{1}{n+1}\int_0^1 t^{n+1}((1-t)^m)' = \frac{m}{n+1}\int_0^1 t^{n+1}(1-t)^{m-1} = \frac{m}{n+1}B_{n+1,m-1}.\end{aligned}$$

Iterating this formula we immediately find

$$B_{n,m} = \frac{m!}{(n+1)\dots(n+m)}B_{n+m,0} = \frac{n!m!}{(n+m)!}\int_0^1 s^{n+m}\,ds = \frac{n!m!}{(n+m+1)!},$$

which gives

$$\int_0^1 t^k(1-t)^k\,dt = B_{k,k} = \frac{(k!)^2}{(2k+1)!}.$$

7. Putting everything together, we have proven

$$\int_{-1}^{1} P_k^2 = \frac{2^{2k+1}(k!)^2}{2k+1}.$$

Let us compute the first three

$$\begin{aligned} P_0(x) &= 1, & \frac{P_0}{\|P_0\|} &= 1/\sqrt{2} = e_0 \\ P_1(x) &= 2x, & \frac{P_1}{\|P_1\|} &= \sqrt{\frac{3}{2}}x = e_1 \\ P_2(x) &= 12x^2-4, & \frac{P_2}{\|P_2\|} &= \frac{\sqrt{5}}{2\sqrt{2}}(3x^2-1) = e_2.\end{aligned}$$

Of course this is no coincidence. There is only one orthonormal basis $\{e_k\}_{k\geq 0}$ of $L^2(-1,1)$ such that each e_k is a polynomial of degree k with positive leading coefficient. These are called the Legendre Polynomials.