The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

3.1. Closed Answer questions.

1. Are $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ isometrically isomorphic as Hilbert spaces?
2. If P_{K_1}, P_{K_2} are the metric projections onto two convex closed sets K_1, K_2 in some Hilbert space H, is it true that $P_{K_1 \cap K_2} = P_{K_1} \circ P_{K_2}$?
3. Given $u \in L^2(0,1)$ there exist a unique polynomial $\bar{p}$ such that $p \mapsto \|u - p\|_{L^2(0,1)}$ is minimal. True or false? **Hint**: Recall that polynomials are dense in $L^2(0,1)$.
4. Show that if K is not convex the metric projection might not exist.

3.2. Projection on subspaces. For each of the following pairs (H, V) where H is an Hilbert space and V is a subspace discuss whether V is closed or not and give a formula for the orthogonal projection $\pi\colon H \to \overline{V}$.

1. $H = L^2(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) < \infty$, and $V = \{u \in H : u \equiv \text{ const. } \mu\text{-a.e.}\}$. **Hint:** you can use the definition, and minimise the function $\mathbb{R} \ni t \mapsto ⨍_\Omega |u(x) - t|^2$.
2. $H = L^2(-1,1)$ and $V = \{u \in H : u(x) = u(-x) \text{ a.e.}\}$.
3. $H = L^2(0,1)$ and $V = \mathbb{R}\log x := \{\alpha \log x : \alpha \in \mathbb{R}\}$. **Hint:** here, as in item 1, V has dimension 1.
4. $H = L^2(\mathbb{R}^3;\mathbb{R}^3)$ and $V = \{\vec{u} \in H : \vec{x} \cdot \vec{u}(x) = 0 \text{ for a.e. } x \in \mathbb{R}^3\}$. **Hint**: everything is happening in the co-domain, so you can use the definition...
5. $(*)$ $H = L^2(\mathbb{R}^d)$ and V is the subspace of radial functions i.e.,
$$V = \{u \in H : \exists U \in L^1_{loc}(0,\infty) \text{ such that } u(x) = U(|x|) \text{ for a.e. } x \in \mathbb{R}^d\}.$$
Hint: work in polar coordinates and use item 1. on each spherical shell.
6. (BONUS) What is the projection of the element $x = \left(\frac{n}{(n+1)^2}\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ onto the subspace generated by $y = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$? **Hint:** the quantity $\|y\|_{\ell^2(\mathbb{N})}$ is a known explicit number. Look it up!

A. ☐ $\left(\frac{\pi^2}{6} - 1\right) y$

B. ☐ $\left(\frac{\pi^2-6}{\sqrt{6}\pi}\right) y$

C. ☐ $\left(1 - \frac{6}{\pi^2}\right) y$

D. ☐ $\left(\frac{\pi^2-\pi}{\sqrt{6}}\right) y$

3.3. Projection on convex sets. For each of the following pairs (H, K) where H is an Hilbert space and K is a convex set (check it, if it is not clear) discuss whether K is closed or not and give a formula for the metric projection $\pi\colon H \to \overline{K}$.

1. $H = L^2(0,1)$ and $K = \{u \in H : u > 0 \text{ a.e.}\}$.
2. $H = L^2(0,2\pi)$ and $K = \{u \in H : u \geq \sin(\cdot) \text{ a.e.}\}$.
3. $H = \mathbb{R}^2$ and $K = [-1,1] \times [-1,1]$.
4. $H = L^2(0,1)$ and $K = \{u \in H : \int_0^1 u\phi \leq 0\}$, where $\phi \in L^2$ is given.

3.4. The tight fishball. Let H be a real Hilbert space and $U \subset H$ be a bounded, nonempty set. Show that, among all the closed balls which contain U, there is only one with minimal radius. **Hint**: look at a sequence of minimizing balls $B_{r_k}(x_k)$ and try to prove that $\{x_k\}$ is Cauchy with the parallelogram identity. It helps to do a picture in 2D, i.e. work out the case $H = \mathbb{R}^2$ first.