The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

3.1. Closed Answer questions.

1. Are $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ isometrically isomorphic as Hilbert spaces?
2. If P_{K_1}, P_{K_2} are the metric projections onto two convex closed sets K_1, K_2 in some Hilbert space H, is it true that $P_{K_1 \cap K_2} = P_{K_1} \circ P_{K_2}$?
3. Given $u \in L^2(0,1)$ there exist a unique polynomial $\bar{p}$ such that $p \mapsto \|u - p\|_{L^2(0,1)}$ is minimal. True or false? **Hint**: Recall that polynomials are dense in $L^2(0,1)$.
4. Show that if K is not convex the metric projection might not exist.

3.2. Projection on subspaces. For each of the following pairs (H, V) where H is an Hilbert space and V is a subspace discuss whether V is closed or not and give a formula for the orthogonal projection $\pi\colon H \to \overline{V}$.

1. $H = L^2(\Omega, \mathcal{F}, \mu)$ with $\mu(\Omega) < \infty$, and $V = \{u \in H : u \equiv \text{ const. } \mu\text{-a.e.}\}$. **Hint:** you can use the definition, and minimise the function $\mathbb{R} \ni t \mapsto \fint_\Omega |u(x) - t|^2$.
2. $H = L^2(-1,1)$ and $V = \{u \in H : u(x) = u(-x) \text{ a.e.}\}$.
3. $H = L^2(0,1)$ and $V = \mathbb{R}\log x \coloneqq \{\alpha \log x : \alpha \in \mathbb{R}\}$. **Hint:** here, as in item 1, V has dimension 1.
4. $H = L^2(\mathbb{R}^3; \mathbb{R}^3)$ and $V = \{\vec{u} \in H : \vec{x} \cdot \vec{u}(x) = 0 \text{ for a.e. } x \in \mathbb{R}^3\}$. **Hint**: everything is happening in the co-domain, so you can use the definition...
5. $(*)$ $H = L^2(\mathbb{R}^d)$ and V is the subspace of radial functions i.e.,
$$V = \{u \in H : \exists U \in L^1_{loc}(0,\infty) \text{ such that } u(x) = U(|x|) \text{ for a.e. } x \in \mathbb{R}^d\}.$$
Hint: work in polar coordinates and use item 1. on each spherical shell.
6. (BONUS) What is the projection of the element $x = \left(\frac{n}{(n+1)^2}\right)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ onto the subspace generated by $y = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$? **Hint:** the quantity $\|y\|_{\ell^2(\mathbb{N})}$ is a known explicit number. Look it up!

A. ☐ $\left(\frac{\pi^2}{6} - 1\right) y$

B. ☐ $\left(\frac{\pi^2 - 6}{\sqrt{6}\pi}\right) y$

C. ☐ $\left(1 - \frac{6}{\pi^2}\right) y$

D. ☐ $\left(\frac{\pi^2-\pi}{\sqrt{6}}\right) y$

3.3. Projection on convex sets. For each of the following pairs (H, K) where H is an Hilbert space and K is a convex set (check it, if it is not clear) discuss whether K is closed or not and give a formula for the metric projection $\pi \colon H \to \overline{K}$.

1. $H = L^2(0,1)$ and $K = \{u \in H : u > 0 \text{ a.e.}\}$.
2. $H = L^2(0,2\pi)$ and $K = \{u \in H : u \geq \sin(\cdot) \text{ a.e.}\}$.
3. $H = \mathbb{R}^2$ and $K = [-1,1] \times [-1,1]$.
4. $H = L^2(0,1)$ and $K = \{u \in H : \int_0^1 u\phi \leq 0\}$, where $\phi \in L^2$ is given.

3.4. The tight fishball. Let H be a real Hilbert space and $U \subset H$ be a bounded, nonempty set. Show that, among all the closed balls which contain U, there is only one with minimal radius. **Hint**: look at a sequence of minimizing balls $B_{r_k}(x_k)$ and try to prove that $\{x_k\}$ is Cauchy with the parallelogram identity. It helps to do a picture in 2D, i.e. work out the case $H = \mathbb{R}^2$ first.

3. Solutions

Solution of 3.1:

1. Yes, they are as any other pair of separable Hilbert spaces. In this case the isometry can be given from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{N})$ as $e_k \to e_{2k}$ if $k \geq 0$ and $e_k \mapsto e_{2k+1}$ if $k < 0$.

2. No this is false, for more or less any pair of nested convex sets K_1, K_2 it is clear that $P_{K_1} \circ P_{K_2} \neq P_{K_2} \circ P_{K_1}$. For example you can try with a disc inscribed in a square (in the plane).

3. No, this is false, the space of polynomials is linear, but not closed (dense, in fact) so we expect to be able to find a counterexample. For example if $u = \exp$ then it is known that it is an analytic function so

$$p_N(x) := \sum_{k=0}^{N} \frac{x^k}{k!} \to \exp \text{ in } L^\infty(0,1),$$

thus this proves that

$$\inf\{\| \exp - p\|_{L^2(0,1)} : p \text{ is a polynomial } \} = 0.$$

If the “inf” was a “min” then exp would be a.e. equivalent to some polynomial, but this is not the case (for example because it $D^\ell \exp$ never vanish, no matter how large ℓ is).

4. If K is not convex the infimum

$$\pi(y) := \inf_{x \in K} \|x - y\|$$

might not be attained by a unique point; take for example $y = 0$ in $\mathbb{R}^2$ and $K = \overline{B_2 \setminus B_1}$.

Solution of 3.2:

1. Constant functions are closed in L^2, by the usual subsequence reasoning. By definition of projection we seek the minimiser of

$$\phi(t) = \fint_\Omega |u(x) - t|^2 \, d\mu = t^2 - 2t \fint_\Omega u \, d\mu + \fint_\Omega u^2 \, d\mu,$$

but a quadratic function of the form $t^2 - 2Bt + C$ reaches its minimum at $\bar{t} = B = \fint_\Omega u \, d\mu$. Thus $\pi(u) = \fint_\Omega u \, d\mu$.

2. The subspace V is just the set of even functions, for which the projection is

$$\pi(u)(x) := u_{\text{even}}(x) = \frac{u(x) + u(-x)}{2} \in V.$$

This is quickly checked by using the characterisation of the projection: if we set $u_{\text{odd}}(x) := (u(x) - u(-x))/2$, then $u = u_{\text{even}} + u_{\text{odd}}$ and for all $\phi \in V$ it holds

$$\langle u - \pi(u), \phi\rangle = \langle u_{\text{odd}}, \phi\rangle = 0.$$

where vanishing the inner product follows from the integrability of the product $u_{\text{odd}}\phi$ and the fact that, as a product of an even and an odd function, it is odd.

3. V is closed (as every finite-dimensional vector subspace). The projection onto a line $\mathbb{R}\xi$ is given by $x \mapsto \langle x, \xi\rangle\xi/\|\xi\|^2$, this is geometrically quite clear and can be proved minimizing $t \mapsto \|x - t\xi\|^2$, as in point 1. Thus in this case $\xi = \log$ and

$$\|\xi\|^2 = \int_0^1 \log(x)^2\, dx \overset{[x=e^t]}{=} \int_{-\infty}^0 t^2 e^t\, dt = -2\int_{-\infty}^0 te^t\, dt = 2\int_{-\infty}^0 e^t\, dt = 2,$$

and so $\pi(u)(x) = \frac{1}{2}\log(x)\int_0^1 u(x')\log(x')\, dx'$.

4. Observe that, for any vector $\vec{u} \in \mathbb{R}^3$, its projection onto the space generated by $\vec{x} \in \mathbb{R}^3$ is given by
$$\vec{u} - (\vec{u}\cdot\vec{x})\vec{x}|\vec{x}|^{-2}$$
Then, if we define
$$\vec{\pi}(u)(x) := \vec{u}(x) - (\vec{u}(x)\cdot\vec{x})\vec{x}|\vec{x}|^{-2},$$
it must also be the L^2 projection, provided it is well-defined. but it is immediate to check that $\vec{\pi}(u) \in L^2(\mathbb{R}^d)$ as $|\vec{\pi}(u)(x)| \leq |\vec{u}(x)|$.

5. We show that V is closed. Take $u_k \in V$, the associated U_k (which are unique) then satisfy

$$\begin{aligned}\|u_k - u_h\|^2_{L^2(\mathbb{R}^d)} &= \sigma(\partial B_1)\int_0^\infty r^{d-1}|U_k(r) - U_h(r)|^2 dr\\ &= \sigma(\partial B_1)\|U_k(r) - U_h(r)\|^2_{L^2((0,\infty), r^{d-1}dr)},\end{aligned}$$

thus $\{U_k\}$ is Cauchy in $L^2((0,\infty), r^{d-1}dr)$ if and only if $\{u_k\}$ is Cauchy in $L^2(\mathbb{R}^d)$. Thus if $u_k \to u$ then $U_k \to U$ and, if the convergence is also a.e. we must have

$$u(x) = \lim_k u_k(x) = \lim_k U(|x|) = U(|x|) \text{ for a.e. } x \in \mathbb{R}^d,$$

thus $u \in V$.

The projection is readily computed, given some $v \in L^2$ we have for each $u \in V$ that, using for each fixed r exercise 1 in the measure space $L^2(\partial B_1, \mathcal{B}(\partial B_1), d\sigma)$,

$$\begin{aligned}\|v - u\|^2_{L^2(\mathbb{R}^d)} &= \int_0^\infty r^{d-1}\, dr \int_{\partial B_1} d\sigma(\theta)|v(r\theta) - U(r)|^2\\ &\geq \int_0^\infty r^{d-1}\, dr \int_{\partial B_1} d\sigma(\theta)\Big|v(r\theta) - \fint_{\partial B_1} v(r\cdot)\, d\sigma\Big|^2.\end{aligned}$$

So if we define $\pi(v)(x) := \fint_{\partial B_1} v(|x|\theta)\, d\sigma(\theta)$ then this computation shows that

$$\|v-u\|^2_{L^2(\mathbb{R}^d)} \geq \|v-\pi(v)\|_{L^2(\mathbb{R}^d)} \text{ for all } u \in V,$$

since $\pi(v) \in V$ we have found the projection. We remark that by changing variables we can rewrite π as

$$\pi(v)(x) = \fint_{\partial B_{|x|}} v(x')\, d\mathcal{H}^{d-1}(x') = \text{average of } v \text{ on the sphere of radius } |x|,$$

where $\mathcal{H}^{d-1}$ is the $d-1$ dimensional Hausdorff measure.

6. The correct answer is C. Indeed, the projection of x onto a one dimensional subspace generated by y is, as in point 1 and 3,

$$\frac{\langle x, y\rangle_{\ell^2}}{\|y\|^2_{\ell^2}} y. \tag{1}$$

We use that

$$\|y\|^2_{\ell^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

(this is known as Basel problem). The inner product is given by

$$\langle x, y\rangle_{\ell^2} = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6} - 1$$

where we used again Basel problem. The projection finally becomes

$$\frac{\langle x, y\rangle_{\ell^2}}{\|y\|^2_{\ell^2}} = \left(\frac{\pi^2}{6} - 1\right)\frac{6}{\pi^2} = 1 - \frac{6}{\pi^2}$$

Solution of 3.3:

1. The closure of K is

$$\overline{K} = \{u \in H : u \geq 0 \text{ a.e.}\}.$$

We define $\pi(u) := u_+ = \max\{u, 0\} \in K$, then $u = u_+ - u_-$ and for all $\phi \in \overline{K}$

$$\langle u - \pi(u), \phi - \pi(u)\rangle = -\int_0^1 u_-(\phi - u_+) = -\int_0^1 \underbrace{u_-\phi}_{\geq 0} - \underbrace{u_-u_+}_{\equiv 0} \leq 0.$$

2. K is closed and the very same reasoning as point 1 gives $\pi(u) = \max\{u, \sin\}$.

3. K is closed (compact, even) and the projection is given by $\pi(x_1, x_2) = (g(x_1), g(x_2))$ where g is the truncation $g\colon s \mapsto \min\{\max\{s, 1\}, -1\}$. We can see this is the projection since we have, for all $z \in K$,

$$\langle x - \pi(x), z - \pi(x)\rangle = (x_1 - g(x_1))(z_1 - g(x_1)) + (x_2 - g(x_2))(z_2 - g(x_2))$$

and we can check directly that

$$(x_i - g(x_i))(z_i - g(x_i)) \begin{cases} = 0 & \text{if } -1 \le x_i \le 1 \\ \le 0 & \text{if } |x_i| > 1 \end{cases}$$

so that the expression above is nonpositive (check separately the cases $x_i > 1$ and $x_i < -1$).

4. Since $K = \varphi^{-1}((-\infty, 0])$, where $\varphi(\eta) = \langle \eta, \phi \rangle$ is a continuous functional, K is closed. We check that

$$\pi(u) = u - \max\{0, \langle u, \tilde{\phi} \rangle\}\tilde{\phi},$$

where $\tilde{\phi} = \phi/\|\phi\|$. If $\langle \psi, \phi \rangle \le 0$ then

$$\begin{aligned} \langle u - \pi(u), \phi - \pi(u) \rangle &= \langle u, \tilde{\phi} \rangle_+ \langle \tilde{\phi}, \psi - u + \langle u, \tilde{\phi} \rangle_+ \tilde{\phi} \rangle \\ &= \underbrace{\langle u, \tilde{\phi} \rangle_+}_{\ge 0} \Big\{ \underbrace{\langle \tilde{\phi}, \psi \rangle}_{\le 0} \underbrace{- \langle u, \tilde{\phi} \rangle + \langle u, \tilde{\phi} \rangle_+}_{\le 0} \Big\} \le 0. \end{aligned}$$

Solution of 3.4: Let $B(x_k, r_k)$ be a minimizing sequence of balls whose closure contains U, that is $U \subseteq \overline{B}(x_k, r_k)$ and

$$r_k \downarrow \varrho := \inf\{r > 0 : \exists x \in H, U \subseteq \overline{B(x, r)}\}.$$

First, $\varrho < \infty$ because U is bounded. The case $\varrho = 0$ is substantially simpler, so assume $\varrho > 0$. The only thing to prove is that (x_k) is Cauchy. Since the radii are monotone we have $U \subset B(x_k, r_k) \cap B(x_h, r_h)$ for all $h \ge k$, and by Pythagoras' theorem (one can work in the euclidean plane generated by x_k, x_h and any element z of U, and make a picture) we find

$$B(x_k, r_k) \cap B(x_h, r_h) \subset B\Big(\frac{x_k + x_h}{2}, (r_k^2 - |x_k - x_h|^2/4)^{1/2}\Big). \tag{2}$$

Thus by definition of ϱ we have $r_k^2 - |x_k - x_h|^2/4 \ge \varrho^2$ for all $h \ge k$, which gives that (x_k) is Cauchy, since $r_k \downarrow \varrho$.

The rigorous proof of (2) is the following. Set $\bar{x} := (x_k + x_h)/2$ and pick any $y \in B(x_k, r_k) \cap B(x_h, r_h)$, then use the parallelogram identity on $[x_k, y, x_h, 2\bar{x} - y]$, which gives

$$4|y - \bar{x}|^2 + |x_k - x_h|^2 = 2|y - x_k|^2 + 2|y - x_h|^2 \le 2r_k^2 + 2r_h^2 \le 4r_k^2,$$

that is (2).