The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

4.1. Closed answer questions.

1. In order to prove that a linear map $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is continuous it is enough to prove that $\|Tu\|_{L^2(\mathbb{R})} \leq 100$, provided $u \in L^2(\mathbb{R})$ and $\|u\|_{L^2(\mathbb{R})} \leq 7$. True or false?
2. Give an example of a nonzero continuous linear functional on $L^2(0,1)$.
3. Alice is given a bounded linear functional $\phi \in L^2(0,1)^*$, and Bob is given a bounded linear functional $\psi \in L^2(0,1)^*$. They check that $\phi(u) = \psi(u)$ for all $u \in C([0,1])$. Is it necessarily true that $\phi = \psi$?
4. If ϕ is a continuous linear functional on an Hilbert space H, then $\ker\phi$ is a closed vector subspace of H. True or false?
5. The inequality "$\|(u_k)\|_{\ell^2(\mathbb{N})} \leq \|(u_k)\|_{\ell^1(\mathbb{N})}$ for all sequences (u_k)", can be equivalently phrased as "the inclusion $\ell^1(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N})$ is 1-Lipschitz". True or false?

4.2. Norm of the multiplication operator. For $u \in H := L^2(0,1)$ consider the operator

$$M_a\colon u(x) \mapsto a(x)u(x),$$

where $a\colon (0,1) \to \mathbb{R}$ is a given measurable function. We want to prove that M_a is continuous from H in itself if and only if $a \in L^\infty(0,1)$, in which case $\|M_a\|_{\mathcal{L}(H)} = \|a\|_{L^\infty(0,1)}$.

1. Given for granted the claim, what is $\|M_{\exp}\|_{\mathcal{L}(H)}$?
2. Prove the inequality

$$\int_0^1 a^2(x)u^2(x)\,dx \leq \sup_{(0,1)} |a|^2 \int_0^1 u^2(x)\,dx,$$

   and deduce that $\|M_a\|_{\mathcal{L}(H)} \leq \|a\|_{L^\infty(0,1)}$.
3. Show that if $E \subset (0,1)$ is measurable with $|E| > 0$, then

$$\frac{\|M_a \mathbf{1}_E\|^2_{L^2(0,1)}}{\|\mathbf{1}_E\|^2_{L^2(0,1)}} = \frac{1}{|E|}\int_E a^2(x)\,dx.$$

4. Choosing properly the measurable set E in the previous point, prove that $\|M_a\|_{\mathcal{L}(H)} \geq \|a\|_{L^\infty}$. **Hint**: try with $E =$"the set where $|a|$ is large" and recall the definition of essential supremum.

4.3. Bounded Linear Operators I. Prove that each of the following linear operators is bounded from $\ell^2(\mathbb{N})$ in itself [1]. Draw the infinite matrix that represents each of them.

1. (Shift operator) $S\colon (u_0, u_1, u_2, \dots) \mapsto (0, u_0, u_1, \dots)$.
2. (Diagonal matrix) $M_\lambda\colon (u_0, u_1, u_2, \dots) \mapsto (\lambda_0 u_0, \lambda_1 u_1, \lambda_2 u_2, \dots)$, where $\{\lambda_j\}_{j\geq 0}$ is some given sequence such that $\sup_{j\geq 0} |\lambda_j| = 7$.
3. $T\colon (u_0, u_1, u_2, \dots) \mapsto (u_0 - u_1, u_1 - u_2, u_2 - u_3, \dots)$.
4. (Hilbert-Schmidt matrix) For each $k \geq 0$ set $(Au)_k := \sum_{j\geq 0} A_{k,j} u_j$, where the infinite matrix $\{A_{i,j}\}_{i\geq 0, j\geq 0}$ satisfies
$$\sum_{i,j} |A_{i,j}|^2 < +\infty.$$
Hint: for each k: $\Big(\sum_{j\geq 0} A_{k,j} u_j\Big)^2 \leq \Big(\sum_{j\geq 0} A_{k,j}^2\Big)\Big(\sum_{j\geq 0} u_j^2\Big)$, by Cauchy Schwarz.

4.4. Bounded or unbounded?. (<u>BONUS</u>) The following operators are well defined bounded operators between Hilbert spaces. True or false?

1. $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $(Tx)_k = \log(1/k) x_k$.
2. $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $(Tx)_k = x_k/(|x_k| + 1)$
3. $T\colon L^2(0,1) \to L^2(0,1)$ given by $Tu = u^2$
4. $T\colon L^2(a,b) \to L^2(a,b)$ given by $Tu = \sqrt{u}$

4.5. Bounded linear operators II. Prove the following inequalities and interpret them as the continuity of a suitable linear map between suitable normed vector spaces:

1. For all $u \in L^2(\mathbb{R})$ it holds
$$\int_0^1 u^2(t)\, dt \leq \int_{\mathbb{R}} u^2(t)\, dt.$$
2. For each polynomial $p(X) = p_0 + p_1 X + \dots + P_k X^N$ it holds
$$\max_{x\in[-1,1]} |p(x)| \leq \sum_{j=0}^{N} |p_j|.$$
3. For all $u \in C^1([0,1])$ with $u(0) = 0$ it holds
$$\max_{x\in[0,1]} |u(x)| \leq \int_0^1 |u'(t)|\, dt.$$
Hint: use the fundamental Theorem of Calculus, i.e., that a function is the integral of its derivative...

[1] Concretely, you have to establish that the ℓ^2 size of the image of any sequence (u_k) is bounded by a multiple of the ℓ^2 size of (u_k) itself.