The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

4.1. Closed answer questions.

1. In order to prove that a linear map $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is continuous it is enough to prove that $\|Tu\|_{L^2(\mathbb{R})} \leq 100$, provided $u \in L^2(\mathbb{R})$ and $\|u\|_{L^2(\mathbb{R})} \leq 7$. True or false?
2. Give an example of a nonzero continuous linear functional on $L^2(0,1)$.
3. Alice is given a bounded linear functional $\phi \in L^2(0,1)^*$, and Bob is given a bounded linear functional $\psi \in L^2(0,1)^*$. They check that $\phi(u) = \psi(u)$ for all $u \in C([0,1])$. Is it necessarily true that $\phi = \psi$?
4. If ϕ is a continuous linear functional on an Hilbert space H, then $\ker \phi$ is a closed vector subspace of H. True or false?
5. The inequality "$\|(u_k)\|_{\ell^2(\mathbb{N})} \leq \|(u_k)\|_{\ell^1(\mathbb{N})}$ for all sequences (u_k)", can be equivalently phrased as "the inclusion $\ell^1(\mathbb{N}) \hookrightarrow \ell^2(\mathbb{N})$ is 1-Lipschitz". True or false?

4.2. Norm of the multiplication operator. For $u \in H := L^2(0,1)$ consider the operator

$$M_a\colon u(x) \mapsto a(x)u(x),$$

where $a\colon (0,1) \to \mathbb{R}$ is a given measurable function. We want to prove that M_a is continuous from H in itself if and only if $a \in L^\infty(0,1)$, in which case $\|M_a\|_{\mathcal{L}(H)} = \|a\|_{L^\infty(0,1)}$.

1. Given for granted the claim, what is $\|M_{\exp}\|_{\mathcal{L}(H)}$?
2. Prove the inequality
$$\int_0^1 a^2(x)u^2(x)\,dx \leq \sup_{(0,1)} |a|^2 \int_0^1 u^2(x)\,dx,$$
and deduce that $\|M_a\|_{\mathcal{L}(H)} \leq \|a\|_{L^\infty(0,1)}$.
3. Show that if $E \subset (0,1)$ is measurable with $|E| > 0$, then
$$\frac{\|M_a \mathbf{1}_E\|^2_{L^2(0,1)}}{\|\mathbf{1}_E\|^2_{L^2(0,1)}} = \frac{1}{|E|}\int_E a^2(x)\,dx.$$
4. Choosing properly the measurable set E in the previous point, prove that $\|M_a\|_{\mathcal{L}(H)} \geq \|a\|_{L^\infty}$. **Hint**: try with $E =$"the set where $|a|$ is large" and recall the definition of essential supremum.

4.3. Bounded Linear Operators I. Prove that each of the following linear operators is bounded from $\ell^2(\mathbb{N})$ in itself [1]. Draw the infinite matrix that represents each of them.

1. (Shift operator) $S\colon (u_0, u_1, u_2, \ldots) \mapsto (0, u_0, u_1, \ldots)$.
2. (Diagonal matrix) $M_\lambda\colon (u_0, u_1, u_2, \ldots) \mapsto (\lambda_0 u_0, \lambda_1 u_1, \lambda_2 u_2, \ldots)$, where $\{\lambda_j\}_{j\geq 0}$ is some given sequence such that $\sup_{j\geq 0} |\lambda_j| = 7$.
3. $T\colon (u_0, u_1, u_2, \ldots) \mapsto (u_0 - u_1, u_1 - u_2, u_2 - u_3, \ldots)$.
4. (Hilbert-Schmidt matrix) For each $k \geq 0$ set $(Au)_k := \sum_{j\geq 0} A_{k,j} u_j$, where the infinite matrix $\{A_{i,j}\}_{i\geq 0, j\geq 0}$ satisfies
$$\sum_{i,j} |A_{i,j}|^2 < +\infty.$$
Hint: for each k: $\Big(\sum_{j\geq 0} A_{k,j} u_j\Big)^2 \leq \Big(\sum_{j\geq 0} A_{k,j}^2\Big)\Big(\sum_{j\geq 0} u_j^2\Big)$, by Cauchy Schwarz.

4.4. Bounded or unbounded?. (BONUS) The following operators are well defined bounded operators between Hilbert spaces. True or false?

1. $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $(Tx)_k = \log(1/k) x_k$.
2. $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $(Tx)_k = x_k/(|x_k| + 1)$
3. $T\colon L^2(0,1) \to L^2(0,1)$ given by $Tu = u^2$
4. $T\colon L^2(a,b) \to L^2(a,b)$ given by $Tu = \sqrt{u}$

4.5. Bounded linear operators II. Prove the following inequalities and interpret them as the continuity of a suitable linear map between suitable normed vector spaces:

1. For all $u \in L^2(\mathbb{R})$ it holds
$$\int_0^1 u^2(t)\, dt \leq \int_{\mathbb{R}} u^2(t)\, dt.$$
2. For each polynomial $p(X) = p_0 + p_1 X + \ldots + P_k X^N$ it holds
$$\max_{x\in[-1,1]} |p(x)| \leq \sum_{j=0}^N |p_j|.$$
3. For all $u \in C^1([0,1])$ with $u(0) = 0$ it holds
$$\max_{x\in[0,1]} |u(x)| \leq \int_0^1 |u'(t)|\, dt.$$
Hint: use the fundamental Theorem of Calculus, i.e., that a function is the integral of its derivative...

[1] Concretely, you have to establish that the ℓ^2 size of the image of any sequence (u_k) is bounded by a multiple of the ℓ^2 size of (u_k) itself.

4. Solutions

Solution of 4.1: 4.1.1. True. If we can prove it then we can prove that T is bounded, exploiting homogeneity. Precisely, if $v \in L^2(\mathbb{R})$ then $u := v/\|v\|$ has norm $\|u\| = 1 < 7$, hence by positive homogeneity

$$\|Tv\|/\|v\| = \|Tu\| \leq 100,$$

this gives $\|Tv\| \leq 100\|v\|$. Since v was arbitrary, this proves that T is bounded.

4.1.2. Take any nonzero $\phi \in L^2$ and consider the map

$$T_\phi(u) \coloneqq \langle u, \phi \rangle_{L^2} = \int_{\mathbb{R}} \phi(x)u(x)dx.$$

This map is continuous by Cauchy–Schwartz and nonzero since

$$T_\phi(\phi) = \|\phi\|^2 > 0.$$

4.1.3. Yes, it is true. In general, two continuous functions that agree on a dense set must agree everywhere. In this case, $C([0,1])$ is dense in $L^2(0,1)$ (with respect to the L^2 topology) and ϕ, ψ are both continuous (being bounded) in the same topology.

4.1.4. True, the pre-image of a closed set via a continuous functions is closed. In this case $\ker \phi = \phi^{-1}(\{0\})$, ϕ is continuous and $\{0\}$ is closed.

4.1.5. True. If we consider the inclusion $\iota\colon \ell^1(\mathbb{N}) \to \ell^2(\mathbb{N})$, then by definition of "operator norm" we have

$$\|\iota\|_{\mathcal{L}(\ell^1(\mathbb{N}),\ell^2(\mathbb{N}))} = \sup_{(u_k)} \frac{\|u\|_{\ell^2(\mathbb{N})}}{\|u\|_{\ell^1(\mathbb{N})}}, \tag{1}$$

so the inequality tells us $\|\iota\|_{\mathcal{L}(\ell^1(\mathbb{N}),\ell^2(\mathbb{N}))} \leq 1$. But then, as remarked in class, the operator norm equals the Lipschitz constant, hence also $\mathrm{Lip}(\iota) \leq 1$.

Conversely, the same remark gives that if $\mathrm{Lip}(\iota) \leq 1$, then $\|\iota\|_{\mathcal{L}(\ell^1(\mathbb{N}),\ell^2(\mathbb{N}))} \leq 1$, then (1) holds, that is the inequality between sequences holds.

Solution of 4.2: 4.2.1. Given for granted the claim

$$\|M_{\exp}\|_{\mathcal{L}(H)} = \|\exp\|_{L^\infty(0,1)} = \sup_{x \in (0,1)} |e^x| = e.$$

4.2.2. This is (a very simple case) of Hölder inequality

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\|g\|_{L^q}, \text{ where } p, q \in [1,\infty], \frac{1}{p} + \frac{1}{q} = 1,$$

with $f = a^2, g = u^2, p = \infty, q = 1$. Alternatively, this inequality arises integrating w.r.t. x the pointwise inequality

$$a^2(x)u^2(x) \leq u^2(x) \sup_{x' \in (0,1)} a^2(x') \text{ for all } x \in (0,1).$$

Since $\|M_a u\|_H^2 = \int_0^1 a^2 u^2$ and $\|u\|_H^2 = \int_0^1 u^2$ then the inequality is saying

$$\frac{\|M_a u\|_H^2}{\|u\|_H^2} \leq \|a^2\|_{L^\infty(0,1)} = \|a\|_{L^\infty(0,1)}^2 \text{ for all } u \in H,$$

which means (by definition) that $\|M_a\|_{\mathcal{L}(H)} \leq \|a\|_{L^\infty}$.

4.2.3. Once again this follows directly computing

$$\|M_a \mathbf{1}_E\|_{L^2(0,1)}^2 = \int_0^1 a^2 \mathbf{1}_E^2 = \int_E a^2,$$
$$\|\mathbf{1}_E\|_{L^2(0,1)}^2 = \int_0^1 \mathbf{1}_E^2 = |E|.$$

4.2.4. Pick any $\mu > 0$ such that $|\{|a| \geq \mu\}| > 0$, then the previous computation with $E := \{|a| > \mu\}$ gives

$$\frac{\|M_a \mathbf{1}_E\|_{L^2(0,1)}^2}{\|\mathbf{1}_E\|_{L^2(0,1)}^2} = \frac{1}{|E|} \int_E \underbrace{a^2(x)}_{\geq \mu^2} \, dx \geq \mu^2,$$

this proves $\|M_a\|_{\mathcal{L}(H)} \geq \mu$. Then we conclude recalling that the essential supremum of a measurable function is defined as the largest such μ we can take:

$$\|a\|_{L^\infty(0,1)} = \operatorname{esssup}|a| = \sup\Big(\{\mu > 0 : |\{|a| \geq \mu\}| > 0\} \cup \{0\}\Big).$$

Solution of 4.3:

4.3.1. The operator is bounded since for any sequence $u \in \ell^2(\mathbb{N})$ we have

$$\|Su\|_{\ell^2(\mathbb{N})}^2 = 0^2 + \sum_{k=0}^{\infty} |u_k|^2 = \sum_{k=0}^{\infty} |u_k|^2 = \|u\|_{\ell^2(\mathbb{N})}^2 .$$

The infinite matrix of S is

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4.3.2. The operator is bounded since for any sequence $u \in \ell^2(\mathbb{N})$ we have

$$\|M_\lambda u\|_{\ell^2(\mathbb{N})}^2 = \sum_{k=1}^{\infty} |\lambda_k u_k|^2 = \sum_{k=1}^{\infty} |\lambda_k|^2 |u_k|^2 \leq \sum_{k=1}^{\infty} 7^2 |u_k|^2 = 49 \, \|u\|_{\ell^2(\mathbb{N})}^2$$

The infinite matrix of M_λ is

$$M_\lambda = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_1 & 0 & 0 & \cdots \\ 0 & 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4.3.3. Note that $\mathbb{C}$ with the Euclidean inner product forms a Hilbert space. This means that operator is bounded since for any sequence $u \in \ell^2(\mathbb{N})$ we can employ the parallelogram identity on $\mathbb{C}$ to compute

$$\begin{aligned} \|Tu\|^2_{\ell^2(\mathbb{N})} &= \sum_{k=1}^{\infty} |u_k - u_{k-1}|^2 \\ &\leq \sum_{k=1}^{\infty} |u_k - u_{k-1}|^2 + |u_k + u_{k-1}|^2 \\ &= \sum_{k=1}^{\infty} 2|u_k|^2 + 2|u_{k-1}|^2 \\ &\leq 4 \sum_{k=0}^{\infty} |u_k|^2 = 4\, \|u\|^2_{\ell^2(\mathbb{N})}. \end{aligned}$$

The infinite matrix of T is

$$T = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4.3.4. To show that the operator is bounded, we take any sequence $u \in \ell^2(\mathbb{N})$ and compute its square norm under the operator A to get

$$\|Au\|^2_{\ell^2(\mathbb{N})} = \sum_{k=0}^{\infty} |(Au)_k|^2 = \sum_{k=0}^{\infty} |\sum_{j=0}^{\infty} A_{kj} u_j|^2.$$

As the hint suggests, we employ the Cauchy-Schwarz inequality to find

$$\sum_{k=0}^{\infty} |\sum_{j=0}^{\infty} A_{kj} u_j|^2 \leq \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} |A_{kj}|^2 \cdot \sum_{j=0}^{\infty} |u_j|^2 \right).$$

Combining the two above equations and taking the square root shows that the map is bounded by

$$\|Au\|_{\ell^2(\mathbb{N})} \leq \left(\sum_{k,j=0}^{\infty} |A_{kj}|^2 \right)^{\frac{1}{2}} \|u\|_{\ell^2(\mathbb{N})}$$

The infinite matrix of A is the matrix itself, i.e.

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} & \cdots \\ A_{10} & A_{11} & A_{12} & A_{13} & \cdots \\ A_{20} & A_{21} & A_{22} & A_{23} & \cdots \\ A_{30} & A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Solution of 4.4:

1. T is not bounded. Indeed, letting $\{e_k\}_{k=1}^{\infty}$ be the standard basis,

$$\|T(e_k)\|_{\ell^2}^2 = \sum_{j=1}^{\infty} |\log(1/j)\delta_{kj}|^2 = \log^2 k.$$

   This proves that $\|T(e_k)\|_{\ell^2} \to \infty$ as $k \to \infty$, which contradicts boundedness.

2. T is bounded, indeed

$$\|Tx\|^2 = \sum_{k=1}^{\infty} |(Tx)_k|^2 = \sum_{k=1}^{\infty} \frac{x_k^2}{(|x_k|+1)^2} \leq \sum_{k=1}^{\infty} x_k^2 = \|x\|^2.$$

3. T is not bounded, indeed $\|Tu\|_{L^2} = +\infty$ for $u(x) = x^{-1/4} \in L^2(0,1)$.
4. T is unbounded, take for instance the sequence $u_n \equiv 1/n$.

Edit: there was a mistake in the solution of 4.4.4. in the version posted on March 27. This is the corrected version.

Solution of 4.5: 4.5.1. The inequality follows from the monotonicity of the integral and $u^2 \geq 0$. It can be interpreted as the fact that the restriction operator

$$\rho \colon L^2(\mathbb{R}) \to L^2(0,1), \quad u \mapsto u|_{(0,1)},$$

(which is linear) is bounded and, more precisely, $\|\rho\|_{\mathcal{L}(L^2(\mathbb{R});L^2(0,1))} \leq 1$.

4.5.2. By the triangular inequality, for each $x \in [-1,1]$ we have

$$|p(x)| \leq \sum_{j=0}^{N} |p_j| \underbrace{|x^j|}_{\leq 1} \leq \sum_{j=0}^{N} |p_j|,$$

so taking the supremum over x we find the sought inequality. We can interpret it as the continuity of the identity map

$$id \colon (V, \|\cdot\|_1) \to (V, \|\cdot\|_{L^\infty}), \quad p \mapsto p,$$

where V is the (infinite dimensional) vector space of polynomials (with real coefficients in one variable), and $\|p\|_1 := \sum_{j=0}^{N} |p_j|$, where $N = N(p)$ is the degree of p.

4.5.3. For all $x \in [0,1]$ and $u \in C^1([0,1])$ with $u(0) = 0$, we have

$$|u(x)| = |u(x) - \underbrace{u(0)}_{=0}| = \left| \int_0^x u'(t)\, dt \right| \underbrace{\leq}_{\triangle} \int_0^x |u'(t)|\, dt \leq \int_0^1 |u'(t)|\, dt,$$

so taking the maximum over x we find the sought estimate. If we define

$$X := \{u \in C^1([0,1]) : \text{ with } u(0) = 0\}, \quad \|u\|_X := \|u'\|_{L^1(0,1)},$$

then the inequality is expressing the continuity of the embedding $X \hookrightarrow C^0([0,1])$. The fact that X is an honest vector space is readily checked, while the fact that $\|\cdot\|_X$ is a norm follows from the linearity of the derivative and

$$\begin{aligned} \|u\|_X = 0 \Longrightarrow u' = 0 \text{ a.e.} &\Longrightarrow u' \equiv 0 \quad (u' \text{ is continuous}) \\ &\Longrightarrow u \equiv \text{ const} \Longrightarrow u \equiv 0 \quad (\text{since } u(0) = 0). \end{aligned}$$