The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

Recall that if $f \in L^1((-\pi,\pi),\mathbb{C})$ and $k \in \mathbb{Z}$ then the k^{th} Fourier coefficient is the complex number defined by

$$c_k(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

5.1. Closed answer questions.

1. Let $V \subset H$ be a closed subspace, $u \in H$, $\tilde{u} \in V$ and assume
$$\langle u-\tilde{u}, v\rangle = 0 \text{ for all } v \in D, \tag{1}$$
where $D \subset V$ is dense. Is it true that $\tilde{u} = P_V(u)$?
2. Is the inclusion map
$$\iota\colon (L^\infty(0,1), \|\cdot\|_{L^1(0,1)}) \to (L^\infty(0,1), \|\cdot\|_{L^2(0,1)}), \quad u \mapsto u,$$
bounded?
3. Let $p(X,Y)$ be a polynomial in two variables and let $f(x) := p(\cos(x),\sin(x))$. If it true that $c_k(f) \neq 0$ only for finitely many values of k (i.e., is f a trigonometric polynomial)? **Hint**: Recall the identity $2\cos(x) = e^{ix} + e^{-ix}$, and use it to express $\cos(x)^m$. Similarly for $\sin(x)$.
4. Compute the Fourier series of $f(x) = e^{-|x|}$ and $g(x) = \sin(x/3)$ (they are not particularly nice, but try to get the computation right!).

5.2. Representation of functionals. (BONUS) For each of the following linear functionals φ defined on an Hilbert space H, determine if it is a continuous linear functional on H and, if so, recall that by Riesz representation theorem $\varphi(x) = \langle x, x_0\rangle_H$ for some $x_0 \in H$. Determine x_0 for every continuous linear functional.

Remark: you get the bonus point if you detect correctly which functionals are not linear continuous and which are, and for the latter give the correct form of x_0.[1]

1. $H = L^2([-\pi,\pi])$ and $\varphi(f) = c_1(f)$
2. $H = L^2([-1,1])$ and $\varphi(f) = f(0)$
3. $H = \ell^2(\mathbb{N},\mathbb{R})$ and $\varphi((x_k)_k) = x_3 + 2x_7$

[1] Careful with the constants!

4. $H = L^2([-1,1])$ and $\varphi(f) = \int_{-1}^{1}(1+f)^2$
5. $H = L^2(\mathbb{R})$ and $\varphi(f) = \frac{1}{3}\int_{-1}^{1} f$
6. $H = \ell^2(\mathbb{N},\mathbb{R})$ and $\varphi((x_k)_k) = \sum_{k=1}^{\infty} \frac{x_k}{k^2}$

5.3. Legendre polynomials III.

1. Using the Stone-Weierstrass Theorem, prove that polynomial functions are dense in $L^2(-1,1)$.
2. Recall the Legendre polynomials $P_k(x) := D^k((x^2-1)^k)$. Show that $\text{span}\{P_k\}_{k\in\mathbb{N}} = \text{span}\{x^k\}_{k\in\mathbb{N}}$, so - by the previous point - Legendre polynomials have dense span. Hence, combining this with exercise 2.4, they form a complete orthogonal system. **Hint**: One inclusion is easy, for the other one show inductively on N that

$$x^N \in \text{span}\{P_0, P_1, \ldots, P_N\}.$$

In order to do so, notice that $P_k(x) = \frac{(2k)!}{k!}x^k + \{\text{lower order terms}\}$, so the leading-order coefficient on P_N is nonzero.

5.4. Fourier series of x^m.

1. Show that $c_k(1) = \sin(\pi k)/(\pi k)$ for all $k \in \mathbb{Z}\setminus\{0\}$. Notice that the identity holds also for all $k \in \mathbb{R}$ (including $k = 0$!).
2. Consider $k \mapsto c_k(f)$ as a function of $k \in \mathbb{R}$, for a fixed function $f \in L^1(-\pi,\pi)$. Show the identity $c_k(xf) = i\frac{d}{dk}c_k(f)$.
3. ($*$) Compute for each $m \in \mathbb{N}$ the Fourier series of x^m. **Hint**: Use the first two points and the analytic expansion $\sin(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell+1}}{(2\ell+1)!}$. Recall that the N^{th} derivative of a function is $N!$ times the N^{th} coefficient of its analytic expansion.