The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

Recall that if $f \in L^1((-\pi,\pi),\mathbb{C})$ and $k \in \mathbb{Z}$ then the k^{th} Fourier coefficient is the complex number defined by

$$c_k(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

5.1. Closed answer questions.

1. Let $V \subset H$ be a closed subspace, $u \in H$, $\tilde{u} \in V$ and assume

$$\langle u - \tilde{u}, v\rangle = 0 \text{ for all } v \in D, \tag{1}$$

   where $D \subset V$ is dense. Is it true that $\tilde{u} = P_V(u)$?

2. Is the inclusion map

$$\iota\colon (L^\infty(0,1), \|\cdot\|_{L^1(0,1)}) \to (L^\infty(0,1), \|\cdot\|_{L^2(0,1)}), \quad u \mapsto u,$$

   bounded?

3. Let $p(X,Y)$ be a polynomial in two variables and let $f(x) := p(\cos(x),\sin(x))$. If it true that $c_k(f) \neq 0$ only for finitely many values of k (i.e., is f a trigonometric polynomial)? **Hint**: Recall the identity $2\cos(x) = e^{ix} + e^{-ix}$, and use it to express $\cos(x)^m$. Similarly for $\sin(x)$.

4. Compute the Fourier series of $f(x) = e^{-|x|}$ and $g(x) = \sin(x/3)$ (they are not particularly nice, but try to get the computation right!).

5.2. Representation of functionals. (BONUS) For each of the following linear functionals φ defined on an Hilbert space H, determine if it is a continuous linear functional on H and, if so, recall that by Riesz representation theorem $\varphi(x) = \langle x, x_0\rangle_H$ for some $x_0 \in H$. Determine x_0 for every continuous linear functional.

Remark: you get the bonus point if you detect correctly which functionals are not linear continuous and which are, and for the latter give the correct form of x_0.[1]

1. $H = L^2([-\pi,\pi])$ and $\varphi(f) = c_1(f)$
2. $H = L^2([-1,1])$ and $\varphi(f) = f(0)$
3. $H = \ell^2(\mathbb{N},\mathbb{R})$ and $\varphi((x_k)_k) = x_3 + 2x_7$

[1] Careful with the constants!

4. $H = L^2([-1,1])$ and $\varphi(f) = \int_{-1}^{1}(1+f)^2$
5. $H = L^2(\mathbb{R})$ and $\varphi(f) = \frac{1}{3}\int_{-1}^{1} f$
6. $H = \ell^2(\mathbb{N}, \mathbb{R})$ and $\varphi((x_k)_k) = \sum_{k=1}^{\infty} \frac{x_k}{k^2}$

5.3. Legendre polynomials III.

1. Using the Stone-Weierstrass Theorem, prove that polynomial functions are dense in $L^2(-1,1)$.
2. Recall the Legendre polynomials $P_k(x) := D^k((x^2-1)^k)$. Show that $\mathrm{span}\{P_k\}_{k\in\mathbb{N}} = \mathrm{span}\{x^k\}_{k\in\mathbb{N}}$, so - by the previous point - Legendre polynomials have dense span. Hence, combining this with exercise 2.4, they form a complete orthogonal system.
Hint: One inclusion is easy, for the other one show inductively on N that
$$x^N \in \mathrm{span}\{P_0, P_1, \dots, P_N\}.$$
In order to do so, notice that $P_k(x) = \frac{(2k)!}{k!}x^k + \{\text{lower order terms}\}$, so the leading-order coefficient on P_N is nonzero.

5.4. Fourier series of x^m.

1. Show that $c_k(1) = \sin(\pi k)/(\pi k)$ for all $k \in \mathbb{Z}\setminus\{0\}$. Notice that the identity holds also for all $k \in \mathbb{R}$ (including $k = 0$!).
2. Consider $k \mapsto c_k(f)$ as a function of $k \in \mathbb{R}$, for a fixed function $f \in L^1(-\pi,\pi)$. Show the identity $c_k(xf) = i\frac{d}{dk}c_k(f)$.
3. $(*)$ Compute for each $m \in \mathbb{N}$ the Fourier series of x^m. **Hint**: Use the first two points and the analytic expansion $\sin(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell z^{2\ell+1}}{(2\ell+1)!}$. Recall that the N^{th} derivative of a function is $N!$ times the N^{th} coefficient of its analytic expansion.

5. Solutions

Solution of 5.1:

5.1.1. The answer is yes. It suffices to check the weak condition of the orthogonal projection operator

$$\langle u - \tilde{u}, v \rangle = 0 \qquad \forall v \in V.$$

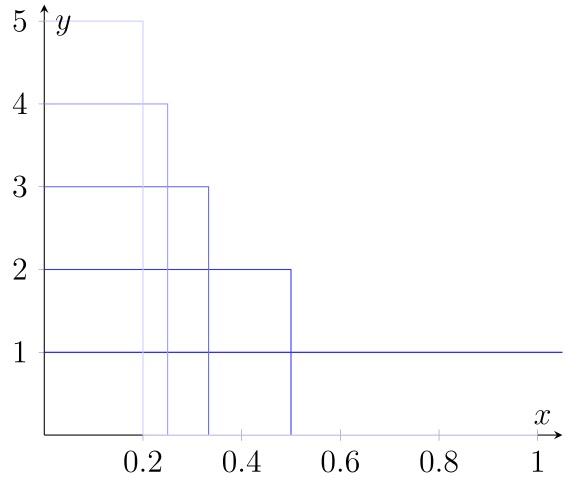
By density of D, for any $v \in V$ there exists a sequence $(v_n)_{n\in\mathbb{N}} \subseteq D$ that converges to v. From the assumption and the continuity of the inner product we find

$$\langle u - \tilde{u}, v \rangle = \lim_{n\to\infty} \langle u - \tilde{u}, v_n \rangle = 0.$$

5.1.2. The answer is no. Define the following sequence of functions

$$f_n := n \cdot \mathbf{1}_{(0,1/n)} \qquad \forall n \geq 1.$$

The first five functions are plotted in the graph below



We then have $\|f_n\|_{L^1(0,1)} = 1$ but $\|f_n\|_{L^2(0,1)} = \sqrt{n}$. This example shows that the operator norm of the inclusion mapping is unbounded. Hence, the inclusion map is not bounded.

5.1.3. The answer is yes. Note that the Fourier coefficient functional is linear, so it suffices to consider a single monomial. Then for any indices $n, m \geq 0$ we employ the binomial theorem to compute

$$\begin{aligned}
\cos(x)^n \sin(x)^m &= \left(\frac{e^{ix} + e^{-ix}}{2}\right)^n \left(-i \cdot \frac{e^{ix} - e^{-ix}}{2}\right)^m \\
&= \frac{(-i)^m}{2^{n+m}} \sum_{k=0}^{n} \left(\binom{n}{k} e^{ikx} e^{-i(n-k)x}\right) \sum_{j=0}^{m} \left(\binom{m}{j} (-1)^{m-j} e^{ijx} e^{-i(m-j)x}\right) \\
&= \frac{(-i)^m}{2^{n+m}} \sum_{k=0}^{n} \sum_{j=0}^{m} \binom{n}{k} \binom{m}{j} (-1)^{m-j} e^{i(2k+2j-n-m)x}.
\end{aligned}$$

This shows that $p(\cos(x), \sin(x))$ can be expressed as a trigonometric polynomial. As the kth-Fourier coefficient is merely a scaled inner product between f and e^{ikx}, we see that $c_k(f)$ will be equal to zero for all but finite many $k \in \mathbb{Z}$.

5.1.4. First, we compute the Fourier coefficients of f.

$$\begin{aligned}
c_k(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-|x|}e^{-ikx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} e^{x}e^{-ikx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-x}e^{-ikx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} e^{(1-ik)x}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-(1+ki)x}\,dx \\
&= \frac{1}{2\pi}\left[\frac{e^{(1-ik)x}}{1-ik}\right]\Bigg|_{-\pi}^{0} + \frac{1}{2\pi}\left[-\frac{e^{-(1+ki)x}}{1+ki}\right]\Bigg|_{0}^{\pi} \\
&= \frac{1}{2\pi}\cdot\frac{1-e^{-(1-ik)\pi}}{1-ik} - \frac{1}{2\pi}\cdot\frac{e^{-(1+ki)\pi}-1}{1+ki} \\
&= \frac{1-e^{-\pi}(-1^k)}{2\pi}\cdot\frac{2}{k^2+1}
\end{aligned}$$

Next compute the Fourier coefficients of g. First observe that

$$\sin(x/3) = -i\cdot\frac{e^{ix/3}-e^{-ix/3}}{2}.$$

Also note that

$$c_k(\overline{g}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overline{g(x)}e^{-ikx}\,dx = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{ikx}\,dx} = \overline{c_{-k}(g)}.$$

Now compute

$$\begin{aligned}
c_k(e^{ix/3}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix/3}e^{-ikx}\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1/3-k)ix}\,dx \\
&= \frac{1}{2\pi}\left[-\frac{e^{(1/3-k)ix}}{1/3-k}\cdot i\right]\Bigg|_{-\pi}^{\pi} \\
&= \frac{(-1)^k\sqrt{3}}{2\pi(1/3-k)}.
\end{aligned}$$

Then use the fact that the kth-Fourier coefficient is linear and the above identity to get

$$\begin{aligned}
c_k(g) &= c_k(\sin(x/3)) \\
&= -i \cdot c_k\left(\frac{e^{ix/3} - e^{-ix/3}}{2}\right) \\
&= -\frac{i}{2} c_k(e^{ix/3}) + \frac{i}{2} c_k(e^{-ix/3}) \\
&= -\frac{i}{2} c_k(e^{ix/3}) + \frac{i}{2} c_k(\ \overline{e^{ix/3}}\ ) \\
&= -\frac{i}{2} c_k(e^{ix/3}) + \frac{i}{2} \overline{c_{-k}(e^{ix/3})} \\
&= -i \cdot \frac{(-1)^k \sqrt{3}}{4\pi(1/3 - k)} + i \cdot \frac{(-1)^k \sqrt{3}}{4\pi(1/3 + k)} \\
&= i \cdot \frac{9k\sqrt{3}}{2\pi(9k^2 - 1)} (-1)^{k+1}.
\end{aligned}$$

The resulting Fourier series has the following shape

Solution of 5.2: Of course, proving that the given functional can be represented as the pairing $\langle \cdot, x_0 \rangle$ for some $x_0 \in H$ automatically proves linearity and continuity (by Cauchy–Schwarz).

1. This is a continuous linear functional since by definition it's given by the L^2 pairing with $x_0 = \frac{1}{2\pi} e^{ix}$.
2. This functional is not even well defined in L^2, as L^2 functions are defined up to sets of measure zero.
3. This is a continuous linear functional given by the ℓ^2 pairing with $x_0 = e_3 + 2e_7$, where $\{e_k\}_{k=1}^{\infty}$ is the standard basis of $\ell^2(\mathbb{N})$.
4. This functional is not linear, since $\varphi(0) \neq 0$.
5. This is a continuous linear functional given by the L^2 pairing with $x_0 = \frac{1}{3}\chi_{[-1,1]}$
6. This is a continuous linear functional given by the ℓ^2 pairing with $x_0 \coloneqq (1/k^2)_{k \in \mathbb{N}} \in \ell^2$

Solution of 5.3: 5.2.1. We use the real version of Stone-Weierstrass, as given in the appendix. Polynomials of one variable are clearly an algebra in $C([-1,1])$ which contains constants. Furthermore, if $a \neq b \in [-1,1]$ then $P(X) = X - a$ vanish at $X = a$ and does not vanish at $X = b$, so it separates points. We conclude that every continuous function can be approximated by polynomials uniformly in $[-1,1]$, hence also in quadratic mean (L^2).

By density of continuous $C_c(-1,1)$ in $L^2(-1,1)$ with respect to the L^2 topology, we conclude that polynomials are dense in $L^2(-1,1)$.

5.2.2. Following the hint we prove by induction that

$$x^N \in \operatorname{span}\{P_0(x), P_1(x), \dots, P_N(x)\}, \text{ for all } N \in \mathbb{N}.$$

The base case is easy since $1 = P_0$ by definition. For the inductive step, we notice that

$$\frac{(2N+2)!}{(N+1)!} x^{N+1} = P_{N+1}(x) - Q_{\le N}(x), \tag{2}$$

where $Q_{\le N}$ is a polynomials of degree at most N, so by inductive assumption

$$Q_{\le N}(x) \in \operatorname{span}\{1, x, \dots, x^N\} \subset \operatorname{span}\{P_0, P_1(x), \dots, P_N(x)\},$$

hence (2) gives $x^N \in \operatorname{span}\{P_0, P_1(x), \dots, P_{N+1}(x)\}$.

Since we checked that $\{P_k\}$ is an orthonormal system in Problem set 2, and now we proved that it has dense span, we have that it is a complete orthogonal system.

Solution of 5.4: 5.3.1. Assume $k \in \mathbb{R} \setminus \{0\}$ and compute

$$c_k(1) = \fint_{-\pi}^{\pi} e^{-ikx}\,dx = \frac{1}{-2i\pi k}[e^{-ikx}]_{-\pi}^{\pi} = \frac{e^{i\pi k} - e^{-i\pi k}}{2i\pi k} = \frac{\sin(\pi k)}{\pi k},$$

where we used $2i\sin(z) = e^{iz} - e^{-iz}$. We remark that $\sin(z)/z$ is analytic in the whole complex plane, so the formula makes perfect sense for $k = 0$ at which the value is 1 (which is the correct one).

5.3.2. This is a direct computation

$$\begin{aligned}\frac{d}{dk} c_k(f) &= \frac{d}{dk} \fint_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = \fint_{-\pi}^{\pi} f(x) \frac{d}{dk}(e^{-ikx}) \\ &= -ik \fint_{-\pi}^{\pi} f(x) e^{-ikx}\,dx = -ik c_k(f),\end{aligned}$$

we can interchange integral and derivative since for each fixed $k \in \mathbb{R}$ we have

$$\sup_{k' \in [k-1, k+1]} |\partial_{k'} e^{-ik'x}| \in L^1_x(-\pi, \pi).$$

5.3.3. Iterating the previous identity and the analytic expansion of $\sin(z)/z$ we find for all $k \in \mathbb{R}$ that

$$\begin{aligned}c_k(x^m) &= i^m \Big(\frac{d}{dk}\Big)^m c_k(1) = i^m \Big(\frac{d}{dk}\Big)^m \frac{\sin(\pi k)}{\pi k} = (\pi i)^m \Big(\frac{d}{dt}\Big)^m \frac{\sin t}{t} \\ &= (\pi i)^m \sum_{j=0}^{m} \binom{m}{j} D^j(\sin t) D^{m-j}(1/t),\end{aligned}$$

where $t = \pi k$. Now we evaluate at an integer k and find

$$\begin{aligned}c_k(x^m) &= (i\pi)^m \sum_{j=0, j \text{ odd}} \binom{m}{j} (-1)^j (-1)(-2)\dots(-m+j)(\pi k)^{-m+j-1} \\ &= (i\pi)^m \sum_{j=0, j \text{ odd}} m! j! (\pi k)^{-m+j-1},\end{aligned}$$

which is, admittedly, not very explicit.