The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with ($*$) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

6.1. Closed answer / quick questions.

1. Let $(H, \langle\cdot,\cdot\rangle)$ be an Hilbert space and $V \subset H$ a proper dense subspace. Can $(V, \langle\cdot,\cdot\rangle)$ be an Hilbert space, at least in some examples?
2. Is the space of sequences with only finitely many nonzero terms, dense in $\ell^2(\mathbb{N})$?
3. Find the Fourier coefficients of $\sin^3(x)$ (compute no integrals!). **Hint**: write $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ and expand the cube.
4. If $f_n \to f$ in $L^1([-\pi,\pi];\mathbb{C})$ then $c_k(f_n) \to c_k(f)$ as $n \to \infty$, uniformly in k? **Hint**: try estimating $|c_k(f_n) - c_k(f)|$ with $\|f_n - f\|_{L^1}$, uniformly in k.

6.2. Fourier coefficients of a shifted function. (BONUS) Let $f\colon \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function such that $f \in L^1((-\pi,\pi),\mathbb{C})$ and let $\tau \in \mathbb{R}$. Define $f_\tau(t) := f(t-\tau)$. Determine the Fourier coefficients of $f_\tau|_{(-\pi,\pi)}$ as a function of the Fourier coefficients of $f|_{(-\pi,\pi)}$.

6.3. Fourier series in $(0,\pi)$. We want to show that every function in $L^2([0,\pi];\mathbb{R})$ can be expressed as a real Fourier series of sines.

1. Show that if $f \in L^2([-\pi,\pi];\mathbb{C})$ is odd then $c_k(f)$ are purely imaginary and $c_0 = 0$;
2. Show that if $f \in L^2([-\pi,\pi];\mathbb{R})$ is odd then its Fourier series simplifies to
$$S_N f(x) = \sum_{1\le k\le N} \underbrace{2ic_k(f)}_{\in\mathbb{R}} \sin(kx)$$
3. Given $g \in L^2([0,\pi];\mathbb{R})$ show that $\tilde{S}_N g \to g$ in L^2 where
$$\tilde{S}_N g(x) := \sum_{1\le k\le N} \tilde{a}_k(g)\sin(kx), \qquad \tilde{a}_k(g) := \frac{2}{\pi}\int_0^\pi g(x)\sin(kx)\,dx \in \mathbb{R}.$$
4. Conclude that $\{\sqrt{2/\pi}\sin(kx)\}_{k\ge1}$ in an Hilbert basis for $L^2([0,\pi];\mathbb{R})$.

6.4. Uniqueness of coefficients in L^1. Fix $f \in L^1([-\pi,\pi];\mathbb{C})$, and let $c_k = c_k(f)$ be its Fourier coefficients, we want to show that if $c_k(f) = 0$ for all $k \in \mathbb{Z}$, then $f \equiv 0$ a.e..

1. Show that if actually $f \in L^2([-\pi,\pi];\mathbb{C})$, then the statement follows directly from a Theorem seen in class.

2. Show that if $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in L^\infty((-\pi,\pi);\mathbb{C})$, then we must have $f = 0$ a.e. Hint: try what happens setting $\phi := \bar{f}/(1+|f|^2)$.
3. Show that if $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in C_c((-\pi,\pi);\mathbb{C})$, then we must have $f = 0$ a.e. **Hint**: we would like to set again $\phi = \bar{f}/(1+|f|^2)$, but f is not continuous... nevertheless $C_c(-\pi,\pi)$ is dense in $L^1(-\pi,\pi)$.
4. Using and appropriate density result seen in class, show that if $c_k(f) = 0$ for all k, then indeed $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in C_c((-\pi,\pi);\mathbb{C})$. Hence by the previous steps $f = 0$.

6.5. Coefficients summability implies convergence. Let $f \in L^1([-\pi,\pi];\mathbb{C})$, and let $c_k = c_k(f)$ be its Fourier coefficients.

1. Show that if $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$, then in fact $f \in L^2([-\pi,\pi];\mathbb{C})$. **Hint**: use Parseval's identity to show that S_N is Cauchy in L^2, then use the prevuious exercise.
2. Show that if $\sum_{k\in\mathbb{Z}} |c_k| < \infty$, then in fact $f \in C_{per}([-\pi,\pi];\mathbb{C})$[1]. **Hint**: show that S_N is Cauchy in the uniform norm, then use the prevuious exercise.

[1]This is a slight abuse of terminology. More precisely: there exist a (necessarily unique) continuous and periodic $\tilde{f}$ such that $\tilde{f} = f$ a.e.