The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

6.1. Closed answer / quick questions.

1. Let $(H, \langle \cdot, \cdot \rangle)$ be an Hilbert space and $V \subset H$ a proper dense subspace. Can $(V, \langle \cdot, \cdot \rangle)$ be an Hilbert space, at least in some examples?
2. Is the space of sequences with only finitely many nonzero terms, dense in $\ell^2(\mathbb{N})$?
3. Find the Fourier coefficients of $\sin^3(x)$ (compute no integrals!). **Hint**: write $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ and expand the cube.
4. If $f_n \to f$ in $L^1([-\pi, \pi]; \mathbb{C})$ then $c_k(f_n) \to c_k(f)$ as $n \to \infty$, uniformly in k? **Hint**: try estimating $|c_k(f_n) - c_k(f)|$ with $\|f_n - f\|_{L^1}$, uniformly in k.

6.2. Fourier coefficients of a shifted function. (BONUS) Let $f \colon \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function such that $f \in L^1((-\pi, \pi), \mathbb{C})$ and let $\tau \in \mathbb{R}$. Define $f_\tau(t) \coloneqq f(t - \tau)$. Determine the Fourier coefficients of $f_\tau|_{(-\pi,\pi)}$ as a function of the Fourier coefficients of $f|_{(-\pi,\pi)}$.

6.3. Fourier series in $(0, \pi)$. We want to show that every function in $L^2([0, \pi]; \mathbb{R})$ can be expressed as a real Fourier series of sines.

1. Show that if $f \in L^2([-\pi, \pi]; \mathbb{C})$ is odd then $c_k(f)$ are purely imaginary and $c_0 = 0$;
2. Show that if $f \in L^2([-\pi, \pi]; \mathbb{R})$ is odd then its Fourier series simplifies to
$$S_N f(x) = \sum_{1 \le k \le N} \underbrace{2i c_k(f)}_{\in \mathbb{R}} \sin(kx)$$
3. Given $g \in L^2([0, \pi]; \mathbb{R})$ show that $\tilde{S}_N g \to g$ in L^2 where
$$\tilde{S}_N g(x) := \sum_{1 \le k \le N} \tilde{a}_k(g) \sin(kx), \qquad \tilde{a}_k(g) := \frac{2}{\pi} \int_0^\pi g(x) \sin(kx)\, dx \in \mathbb{R}.$$
4. Conclude that $\{\sqrt{2/\pi} \sin(kx)\}_{k \ge 1}$ in an Hilbert basis for $L^2([0, \pi]; \mathbb{R})$.

6.4. Uniqueness of coefficients in L^1. Fix $f \in L^1([-\pi, \pi]; \mathbb{C})$, and let $c_k = c_k(f)$ be its Fourier coefficients, we want to show that if $c_k(f) = 0$ for all $k \in \mathbb{Z}$, then $f \equiv 0$ a.e..

1. Show that if actually $f \in L^2([-\pi, \pi]; \mathbb{C})$, then the statement follows directly from a Theorem seen in class.

2. Show that if $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in L^\infty((-\pi,\pi);\mathbb{C})$, then we must have $f = 0$ a.e. Hint: try what happens setting $\phi := \bar{f}/(1+|f|^2)$.

3. Show that if $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in C_c((-\pi,\pi);\mathbb{C})$, then we must have $f = 0$ a.e. **Hint**: we would like to set again $\phi = \bar{f}/(1+|f|^2)$, but f is not continuous... nevertheless $C_c(-\pi,\pi)$ is dense in $L^1(-\pi,\pi)$.

4. Using and appropriate density result seen in class, show that if $c_k(f) = 0$ for all k, then indeed $\int_{-\pi}^{\pi} f\phi = 0$ for all $\phi \in C_c((-\pi,\pi);\mathbb{C})$. Hence by the previous steps $f = 0$.

6.5. Coefficients summability implies convergence. Let $f \in L^1([-\pi,\pi];\mathbb{C})$, and let $c_k = c_k(f)$ be its Fourier coefficients.

1. Show that if $\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty$, then in fact $f \in L^2([-\pi,\pi];\mathbb{C})$. **Hint**: use Parseval's identity to show that S_N is Cauchy in L^2, then use the prevuious exercise.

2. Show that if $\sum_{k\in\mathbb{Z}} |c_k| < \infty$, then in fact $f \in C_{per}([-\pi,\pi];\mathbb{C})$[1]. **Hint**: show that S_N is Cauchy in the uniform norm, then use the prevuious exercise.

[1]This is a slight abuse of terminology. More precisely: there exist a (necessarily unique) continuous and periodic $\tilde{f}$ such that $\tilde{f} = f$ a.e.

6. Solutions

Solution of 6.1:

1. No V cannot be complete. Pick any $x \in H \setminus V$ and, by density, a sequence $\{v_k\} \subset V$ such that $v_k \to x$ in H. Since converging sequences are Cauchy, we find that — if V was Hilbert (complete) — the limit point (which is x by uniqueness) should lie in V, but this is impossible by construction.

2. Yes, it is. Given any $f \in \ell^2(\mathbb{N})$ consider the truncated sequence $\{f_N\} \subset \ell^2(\mathbb{N})$ defined by
$$f_N(k) = \begin{cases} f(k) & k \le N \\ 0 & k > N. \end{cases}$$
By definition
$$\|f - f_N\|_{\ell^2}^2 = \sum_{k>N} |f(k)|^2 \to 0 \text{ as } N \to \infty$$
since it is the tail of a convergent sum.

3. Expanding the cube one finds that
$$(\sin x)^3 = \frac{i}{8}\left(e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}\right)$$
so the only nonzero Fourier coefficients are
$$c_{\pm 3}(\sin^3) = \frac{\pm i}{8}, \quad c_{\pm 1}(\sin^3) = \frac{\mp 3i}{8}.$$
(We are using that Fourier coefficients are unique!)

4. Yes, this is true. We estimate
$$|c_k(f) - c_k(f_n)| = |c_k(f - f_n)| \le \|e^{-ikx}\|_{L^\infty(-\pi,\pi)} \|f - f_n\|_{L^1(-\pi,\pi)} \le \|f - f_n\|_{L^1(-\pi,\pi)}.$$
And the right hand side is uniform in k and infinitesimal as $n \to \infty$.

Solution of 6.2: Using that both f and $e^{ik(\cdot)}$ are 2π-periodic, we find
$$\begin{aligned} c_k(f_\tau) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\tau(x) e^{-ikx} dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\tau) e^{-ikx} dx \\ [x' = x - \tau] \qquad &= \frac{1}{2\pi}\int_{-\pi-\tau}^{\pi-\tau} f(x') e^{-ik(x'+\tau)} dx' \\ &= \frac{e^{-ik\tau}}{2\pi}\int_{-\pi}^{\pi} f(x') e^{-ikx'} dx' \\ &= e^{-ik\tau} c_k(f) \end{aligned}$$

Solution of 6.3: 1. Let $f \in L^2([-\pi,\pi];\mathbb{C})$ be odd. Given $k \in \mathbb{Z}$ compute

$$\begin{aligned} c_k(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\ dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-ikx}\ dx + \frac{1}{2\pi}\int_{-\pi}^{0} f(x)e^{-ikx}\ dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x)e^{-ikx}\ dx + \frac{1}{2\pi}\int_{0}^{\pi} f(-x)e^{ikx}\ dx \\ &= \frac{1}{2\pi}\int_{0}^{\pi} f(x)(e^{-ikx} - e^{ikx})\ dx \\ &= \frac{-i}{\pi}\int_{0}^{\pi} f(x)\sin(kx)\ dx, \end{aligned}$$

where we used the change of variables $x \mapsto -x$. Since $(f\sin) \in \mathbb{R}$, we conclude $c_k(f) \in i\mathbb{R}$. Furthermore, we see that $c_0(f) = 0$.

2. Note that for $k \in \mathbb{Z}$ we have

$$\begin{aligned} c_{-k}(f) = &= \frac{-i}{\pi}\int_{0}^{\pi} f(x)\sin(-kx)\ dx \\ &= \frac{i}{\pi}\int_{0}^{\pi} f(x)\sin(kx)\ dx \\ &= -c_k(f). \end{aligned}$$

Writing $e^{i\vartheta} = \cos(\vartheta) + i\sin(\vartheta)$, we obtain

$$\begin{aligned} S_N(f) &= \sum_{k=1}^{N}[c_k(f)\cos(kx) + ic_k(f)\sin(kx)] \\ &\quad + \sum_{k=1}^{N}[c_{-k}(f)\cos(-kx) + ic_{-k}(f)\sin(-kx)] \\ &= \sum_{k=1}^{N}[c_k(f)\cos(kx) + ic_k(f)\sin(kx)] \\ &\quad + \sum_{k=1}^{N}[-c_k(f)\cos(kx) + ic_k(f)\sin(kx)] \\ &= \sum_{k=1}^{N} 2ic_k(f)\sin(kx), \end{aligned}$$

where we used that $\sin(\cdot)$ is odd and $\cos(\cdot)$ is even.

3. Let $f : [-\pi,\pi] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} g(x), & \text{if } x \in [0,\pi] \\ -g(-x), & \text{if } x \in [-\pi,0). \end{cases}$$

Then $f \in L^2([-\pi,\pi];\mathbb{R})$ is odd and we may use (1.) and (2.) to compute the N-th partial sum of the Fourier series

$$S_N f(x) = \sum_{k=1}^{N} \tilde{a}_k(g)\sin(kx),$$

for a.e. $x \in [-\pi,\pi]$. Applying Corollary 2.7 we conclude the convergence

$$\begin{aligned}\left\| \tilde{S}_N g - g \right\|_{L^2(0,\pi)} &= \|S_N f - f\|_{L^2(0,\pi)} \\ &\leq \|S_N f - f\|_{L^2(-\pi,\pi)} \to 0,\end{aligned}$$

as $N \to \infty$.

4. In 3. we saw that $\operatorname{Span}\left\{\sqrt{2/\pi}\sin(kx)\right\}_{k\geq 1}$ is dense in $L^2([0,\pi];\mathbb{R})$. It remains to show L^2-orthonormality. Recall the identity

$$\sin(x)\sin(y) = \frac{1}{2}(\cos(x-y) - \cos(x+y)).$$

For $k \neq j \geq 1$ compute

$$\begin{aligned}\left\langle \sqrt{2/\pi}\sin(kx), \sqrt{2/\pi}\sin(jx) \right\rangle_{L^2} &= \frac{2}{\pi}\int_0^{\pi} \sin(kx)\sin(jx)\ dx \\ &= \frac{1}{\pi}\int_0^{\pi} \cos((k-j)x) - \cos((k+j)x)\ dx \\ &= \frac{1}{\pi}\left(\left[\frac{1}{k-j}\sin((k-j)x)\right]_0^{\pi} + \left[\frac{1}{k+j}\sin((k+j)x)\right]_0^{\pi}\right) \\ &= 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\left\| \sqrt{2/\pi}\sin(kx) \right\|_{L^2}^2 &= \frac{2}{\pi}\int_0^{\pi} \sin(kx)^2\ dx \\ &= \frac{1}{\pi}\int_0^{\pi} 1 - \cos(2kx)\ dx \\ &= \frac{1}{\pi}\left(\pi - \left[\frac{1}{2k}\sin(2kx)\right]_0^{\pi}\right) \\ &= 1.\end{aligned}$$

Solution of 6.4:

1. We can use Parseval's Identity and get that f has zero norm:

$$||f||_{L^2}^2 = 2\pi \cdot \sum_k |c_k(f)|^2 = 0.$$

This shows that f has to be equal to 0 almost everywhere.

2. Using $\phi = \frac{\bar{f}}{1+|f|^2}$, which is bounded by construction, we get

$$0 = \int_{-\pi}^{\pi} f\phi = \int_{-\pi}^{\pi} f \cdot \frac{\bar{f}}{1+|f|^2} = \int_{-\pi}^{\pi} \frac{|f|^2}{1+|f|^2},$$

since the integrand is nonnegative, we must have $f = 0$ almost everywhere.

3. As in the Solution of Problem 3.1.2, we use mollifiers ρ_ε and note that $\rho_\varepsilon * f \to f$ in $L^1((-\pi,\pi);\mathbb{C})$. Thus there exists a sequence ε_j such that $\left(\rho_{\varepsilon_j} * f\right)(x) \to f(x)$ for a.e. $x \in (-\pi,\pi)$. But for any such x:

$$f(x) = \lim_{j\to\infty} \left(\rho_{\varepsilon_j} * f\right)(x) = \lim_{j\to\infty} \int_{-\pi}^{\pi} \rho_{\varepsilon_j}(x-y) f(y)\ \mathrm{d}y = 0,$$

since $\rho_{\varepsilon_j}(x-\cdot) \in C_c^1((-\pi,\pi);\mathbb{C})$ for ε_j small enough.

Alterantively, pick a sequence $\{f_n\} \subset C_c(-\pi,\pi)$ such that $f_n \to f$ in a.e.. Then test the inequality with $\phi_n := \bar{f}_n/(1+|f_n|^2)$: by the dominated convergence theorem we find, as in the previous point,

$$0 = \lim_n \int_{-\pi}^{\pi} \frac{f\bar{f}_n}{1+|f_n|^2} = \int_{-\pi}^{\pi} \frac{|f|^2}{1+|f|^2}.$$

4. Let $\phi \in C_c^1((-\pi,\pi);\mathbb{C})$. Since ϕ is compactly supported, we can extend it periodically to all of $\mathbb{R}$ and view it as a continuous 2π-periodic function. By the Stone-Weierstrass Theorem, there exist a sequence of trigonometric polynomials that converge to ϕ in the $L^\infty(-\pi,\pi)$ norm.

Let now $\varepsilon > 0$ and choose some trigonometric polynomial $p = \sum_{|k|\le N} p_k e^{ikx}$ such that $||\phi - p||_\infty < \varepsilon$.

Then we can estimate:

$$\begin{aligned}\left|\int_{-\pi}^{\pi} f\phi\ \mathrm{d}x\right| &\le \left|\int_{-\pi}^{\pi} f\cdot p\ \mathrm{d}x\right| + \left|\int_{-\pi}^{\pi} f\cdot(\phi-p)\ \mathrm{d}x\right| \\ &\le \sum_{|k|\le N} |p_k c_k(f)| + \varepsilon||f||_1 = \varepsilon||f||_1\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\int_{-\pi}^{\pi} f\phi\ \mathrm{d}x = 0$

Solution of 6.5: (1.) Let $M > N$ and observe that

$$\|S_M - S_N\|_{L^2} = \sum_{N\le|k|\le M} |c_k|^2 \to 0$$

as $N, M \to \infty$, since it's the tail of a convergent series. Here we used that the Fourier coefficients of $S_M - S_N \in L^2$ are trivially given by c_k for $N \le |k| \le M$ and 0 otherwise, along with Parseval's identity. Thus, S_N has an L^2 limit $\tilde{f}$ and $c_k(f) = c_k(\tilde{f})$ for every k.

By the first point of the previous exercise, since $c_k(f - \tilde{f}) = 0$ for every k it holds $f = \tilde{f}$ as L^1 functions.

(2.) The solution is similar to the previous point, with the only difference that now we check that $\{S_N\}$ is Cauchy in the uniform norm.

$$\|S_M - S_N\|_\infty \leq \sum_{N \leq |k| \leq M} |c_k| \to 0$$

as $N, M \to \infty$, since it's again the tail of a convergent series. The conclusion follows as before.