The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with ($*$) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

7.1. Closed answer questions.

1. Construct $f\colon [-\pi,\pi] \to \mathbb{R}$ which is continuous, but not Hölder at $\bar{x}=0$. **Hint**: try with $1/\log(t)$.
2. Let V be the vector space of sequences $f\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ such that
$$\|f\|_V := \Big\{ \sum_{k\geq 1} k^2 |f(k)|^2 \Big\}^{1/2} < \infty.$$
Can you choose a scalar product on V that makes V an Hilbert space? **Hint**: try to construct an L^2 space over $\mathbb{N}$ with the right measure.
3. Let V be the vector space of sequences $f\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ such that
$$\|f\|_V := \sum_{k\geq 1} k|f(k)| < \infty.$$
Can you choose a scalar product on V that makes V an Hilbert space?
4. Explain the difference between the following spaces of (real) functions and provide elements that fit in one but none of the others:
$$C_{per}([-\pi,\pi]), \quad C^2_{per}([-\pi,\pi]), \quad C((-\pi,\pi)), \quad C([-\pi,\pi]).$$

7.2. Fourier series convergence recap. For each of the following functions defined on $[-\pi,\pi]$,

- $f_1(x) = \tan(\sin(x))$
- $f_2(x) = |x|^{2/3}$
- $f_3(x) = x$
- $f_4(x) = e^{-x^2}$
- $f_5(x) = (x^2-\pi^2)^2$

answer to the following questions using the Theorems seen in class. If you cannot apply any of those Theorems, that's still a valid answer!

1. Are the Fourier coefficients well defined?
2. Is it true that $S_N(f) \to f$ in L^2?

3. Is it true that $S_N(f)(x) \to f(x)$ for all $x \in (-\pi, \pi)$? What about $x = \pm\pi$? **Hint**: Recall Theorem 2.27.
4. Is it true that $S_N(f) \to f$ in C_{per}? **Hint**: Recall Corollary 2.20.
5. If possible, give two non-negative values of $0 \le \alpha_1 < \alpha_2$ such that

$$\sum_{k\in\mathbb{Z}} |k|^{\alpha_1} |c_k(f)| < +\infty, \text{ but } \sum_{k\in\mathbb{Z}} |k|^{\alpha_2} |c_k(f)| = +\infty.$$

Hint: Recall Theorems 2.22 and 2.25.

(BONUS): Answer questions 1,2,3 and 4 for the function $f_6(x) = |x|^{-1/2}$.

7.3. The Dirichlet kernel is not in L^1. Recall that $D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}$, for all $n \ge 1$ and $x \in \mathbb{R}$, is a 2π periodic function.

1. Using $|\sin(t)| \le |t|$, then changing variables and then dividing the domain of integration, show that

$$\int_0^\pi |D_n(x)|\, dx > 2 \sum_{j=0}^{n-1} \int_{j\pi}^{(j+1)\pi} |\sin(y)| \frac{dy}{y}.$$

2. Show that for each $j \ge 0$ it holds

$$\int_{j\pi}^{(j+1)\pi} |\sin(y)| \frac{dy}{y} \ge \frac{c}{j+1},$$

for some (explicit) constant $c > 0$.

3. Conclude that $\|D_n\|_{L^1(0,\pi)} \ge O(\log n)$ as $n \to \infty$. **Hint**: Recall the asymptotic behaviour of the harmonic series: $H_n := \sum_{k=1}^N 1/k \asymp \log n$.

7.4. Fourier series of the product. Let $f, g \in L^2([-\pi,\pi];\mathbb{C})$, prove that

$$c_k(fg) = \sum_{j\in\mathbb{Z}} c_j(f) c_{k-j}(g) \quad \text{for all } k \in \mathbb{Z},$$

and in particular that $c_k(fg)$ is well-defined, and that the series at the right-hand side is absolutely convergent. **Hint 1**: First, show the formula for $S_N(f)$ and $S_N(g)$. Justify the limit carefully, you need no more than the dominated convergence and Cauchy–Schwarz. **Hint 2**: Recall that if $S_N(f) \to f$ and $S_N(g) \to g$ in L^2, then $S_N(f)S_N(g) \to fg$ in L^1. This implies that $c_k(S_N(f)S_N(g)) \to c_k(fg)$ (for instance, by exercise 1.4 in Problem set 6).