The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

7.1. Closed answer questions.

1. Construct $f\colon [-\pi,\pi] \to \mathbb{R}$ which is continuous, but not Hölder at $\bar{x} = 0$. **Hint**: try with $1/\log(t)$.
2. Let V be the vector space of sequences $f\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ such that
$$\|f\|_V := \Big\{ \sum_{k\geq 1} k^2 |f(k)|^2 \Big\}^{1/2} < \infty.$$
Can you choose a scalar product on V that makes V an Hilbert space? **Hint**: try to construct an L^2 space over $\mathbb{N}$ with the right measure.
3. Let V be the vector space of sequences $f\colon \mathbb{N}\setminus\{0\} \to \mathbb{R}$ such that
$$\|f\|_V := \sum_{k\geq 1} k|f(k)| < \infty.$$
Can you choose a scalar product on V that makes V an Hilbert space?
4. Explain the difference between the following spaces of (real) functions and provide elements that fit in one but none of the others:
$$C_{per}([-\pi,\pi]), \quad C^2_{per}([-\pi,\pi]), \quad C((-\pi,\pi)), \quad C([-\pi,\pi]).$$

7.2. Fourier series convergence recap. For each of the following functions defined on $[-\pi,\pi]$,

- $f_1(x) = \tan(\sin(x))$
- $f_2(x) = |x|^{2/3}$
- $f_3(x) = x$
- $f_4(x) = e^{-x^2}$
- $f_5(x) = (x^2-\pi^2)^2$

answer to the following questions using the Theorems seen in class. If you cannot apply any of those Theorems, that's still a valid answer!

1. Are the Fourier coefficients well defined?
2. Is it true that $S_N(f) \to f$ in L^2?

3. Is it true that $S_N(f)(x) \to f(x)$ for all $x \in (-\pi, \pi)$? What about $x = \pm\pi$? **Hint**: Recall Theorem 2.27.
4. Is it true that $S_N(f) \to f$ in C_{per}? **Hint**: Recall Corollary 2.20.
5. If possible, give two non-negative values of $0 \leq \alpha_1 < \alpha_2$ such that

$$\sum_{k\in\mathbb{Z}} |k|^{\alpha_1}|c_k(f)| < +\infty, \text{ but } \sum_{k\in\mathbb{Z}} |k|^{\alpha_2}|c_k(f)| = +\infty.$$

   Hint: Recall Theorems 2.22 and 2.25.

(BONUS): Answer questions 1,2,3 and 4 for the function $f_6(x) = |x|^{-1/2}$.

7.3. The Dirichlet kernel is not in L^1. Recall that $D_n(x) = \frac{\sin((n+1/2)x)}{\sin(x/2)}$, for all $n \geq 1$ and $x \in \mathbb{R}$, is a 2π periodic function.

1. Using $|\sin(t)| \leq |t|$, then changing variables and then dividing the domain of integration, show that

$$\int_0^\pi |D_n(x)|\,dx > 2\sum_{j=0}^{n-1}\int_{j\pi}^{(j+1)\pi} |\sin(y)|\frac{dy}{y}.$$

2. Show that for each $j \geq 0$ it holds

$$\int_{j\pi}^{(j+1)\pi} |\sin(y)|\frac{dy}{y} \geq \frac{c}{j+1},$$

   for some (explicit) constant $c > 0$.

3. Conclude that $\|D_n\|_{L^1(0,\pi)} \geq O(\log n)$ as $n \to \infty$. **Hint**: Recall the asymptotic behaviour of the harmonic series: $H_n \coloneqq \sum_{k=1}^N 1/k \asymp \log n$.

7.4. Fourier series of the product. Let $f, g \in L^2([-\pi,\pi];\mathbb{C})$, prove that

$$c_k(fg) = \sum_{j\in\mathbb{Z}} c_j(f)c_{k-j}(g) \quad \text{for all } k \in \mathbb{Z},$$

and in particular that $c_k(fg)$ is well-defined, and that the series at the right-hand side is absolutely convergent. **Hint 1**: First, show the formula for $S_N(f)$ and $S_N(g)$. Justify the limit carefully, you need no more than the dominated convergence and Cauchy–Schwarz. **Hint 2**: Recall that if $S_N(f) \to f$ and $S_N(g) \to g$ in L^2, then $S_N(f)S_N(g) \to fg$ in L^1. This implies that $c_k(S_N(f)S_N(g)) \to c_k(fg)$ (for instance, by exercise 1.4 in Problem set 6).

7. Solutions

Solution of 7.1:

1. Let $f(t) = 1/\log(t)$ for $t \in (0, \pi]$ and $f(0) = 0$. Then, f is continuous in $[0, \pi]$ and suppose by contradiction that is also Hölder continuous in $[0, \pi]$; this means that there exists $\alpha \in (0, 1)$, $C > 0$ such that
$$\sup_{x \neq y \in [0,\pi]} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq C.$$
Setting $y = 0$ we obtain for any $x \in (0, \pi]$ that
$$0 < \frac{1}{C} \leq \frac{x^\alpha}{|f(x)|} = |\log(x/2\pi)|x^\alpha,$$
which is in contradiction with the fact that $\lim_{x \searrow 0} |\log(x)|x^\alpha = 0$.

2. Consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where
$$\mu(A) = \sum_{k \in \mathbb{N}} k \mathbb{1}_A(k).$$
By definition
$$\int_{\mathbb{N}} |f|^2 \, d\mu = \sum_{k \in \mathbb{N}} k^2 |f(k)|^2 = \|f\|_V^2,$$
that is, $\|\cdot\|_V$ is a norm associated to an L^2 space, thus arising from an inner product. We conclude that V is a Hilbert space.

3. As in the previous point we have
$$\|f\|_V = \|f\|_{L^1(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)},$$
where μ is as above. Now, the L^1-norm is not induced by an inner product and hence V endowed with the norm $\|\cdot\|_V$ has not a Hilbert space structure. To show this, we check that the parallelogram identity is not satisfied: consider $f = (1, 1, 0, \dots)$ and $g = (1, -1, 0, \dots)$, then
$$\begin{aligned} &\|f + g\|_V^2 + \|f - g\|_V^2 = (2 + 0)^2 + (0 + 4)^2 = 20 \\ &\neq 36 = 2 \cdot (1 + 2)^2 + 2 \cdot (1 + 2)^2 = 2\|f\|_V^2 + 2\|g\|_V^2. \end{aligned}$$

4. Recall the definitions
$$\begin{aligned} C((-\pi, \pi)) &= \{ \text{ real-valued continuous functions on } (-\pi, \pi)\}, \\ C([-\pi, \pi]) &= \{ \text{ real-valued continuous functions on } [-\pi, \pi]\}, \\ C_{per}([-\pi, \pi]) &= \{f \in C([-\pi, \pi]) : f(-\pi) = f(\pi)\}, \\ C^2_{per}([-\pi, \pi]) &= \{f \in C([-\pi, \pi]) : f \text{ is twice continuous differentiable} \\ &\qquad \text{with } f, f', f'' \in C_{per}([-\pi, \pi])\}. \end{aligned}$$

From the definitions the following inclusions follow immediatly

$$C^2_{per}([-\pi,\pi]) \subset C_{per}([-\pi,\pi]) \subset C([-\pi,\pi]) \subset C((-\pi,\pi)).$$

We claim that all inclusions are strict. For every inclusion we construct functions that belong to the larger space but not to the smaller one. Let $f, g, h : [-\pi,\pi] \to \mathbb{R}$ be given by

$$\begin{aligned} f(x) &= (\pi - x)^{-1}, \\ g(x) &= x, \\ h(x) &= |x|. \end{aligned}$$

Then

$$\begin{aligned} &f \in C((-\pi,\pi)) \backslash C([-\pi,\pi]), \\ &g \in C([-\pi,\pi]) \backslash C_{per}([-\pi,\pi]), \\ &h \in C_{per}([-\pi,\pi]) \backslash C^2_{per}([-\pi,\pi]). \end{aligned}$$

Solution of 7.2:

1. All functions f_k, $k = 1, \dots, 6$ are $L^1(-\pi,\pi)$, so the Fourier coefficients are well defined.

2. This L^2 convergence holds for all functions that are of class L^2, thus it is valid for f_1, f_2, f_3, f_4, f_5. It is not valid for f_6, since $f_6 \notin L^2$ (if the convergence was true, it would imply that $f_6 \in L^2$).

3. For f_1, f_4 and f_5 the convergence is uniform (thus pointwise) since they are continuous with piecewise continuous derivatives (including extrema!). For f_2 pointwise convergence still holds, since f is Hölder continuous in $[-\pi,\pi]$. As per f_3, the function is clearly C^1 in the interior of $(-\pi,\pi)$ so we have pointwise convergence there. We cannot possibly have pointwise convergence at $x = \pm\pi$, simply because $f_3(\pi) \neq f_3(-\pi)$, but $S_N(f_3)(\pi) = S_N(f_3)(-\pi)$ for all N since $\{S_N(f)\}_N$ are 2π periodic functions. Finally, for f_6 we have pointwise convergence only outside of 0, since the limit is not defined in 0 and the function is locally Lipschitz outside of the origin.

4. For f_1, f_4, f_5 we already observed that the convergence is uniform. Since all these functions are continuous on periodic, we can say that the convergence happens in C_{per}. The function f_2 is not piecewise C^1, since it cannot be differentiated in the interval $[0,\pi]$, thus we cannot apply any of the results we have seen that ensure uniform convergence. On the other hand there is no obvious contradiction in the fact that $S_N(f_2) \to f_2$ uniformly. Thus in this case we cannot apply our results directly, and this is a correct answer for the sake of the exercise. Finally, since f_3 and f_6 are not continuous and periodic, they cannot be approximated uniformly with their partial Fourier sums (the partial Fourier sums lie in C_{per} and uniform limit of C_{per} functions lies in C_{per}).

5. We know $f_1 \in C^\infty(\mathbb{R})$ and and periodic, thus $\sum_{k\in\mathbb{Z}}|k|^{\alpha_1}|c_k(f)| < \infty$ for all $\alpha_1 \geq 0$. Thus no α_2 can be found. For f_2, we know $f_2 \notin C^1_{per}$ and thus it must be that $\sum_{k\in\mathbb{Z}}|k|^2|c_k(f_2)| = \infty$, since if this series was finite it would imply $S_N(f_2) \to f_2$ in C^2_{per}, but the limit is not even differentiable one time! Thus $\alpha_2 = 2$ works. Concerning α_1 our Theorems do not grant that even $\alpha_1 = 0$ works (which remember, it's a viable answer!). For f_3, we have that $\sum_{k\in\mathbb{Z}}|c_k(f)| = \infty$, otherwise f would be continuous and periodic. Thus $\alpha_2 = 0$ and no α_1 can be found. Next, note that f_4 is piecewise C^1 but not C^2_{per}. Then, we can check that for instance $\sum_{k\in\mathbb{Z}}|k|^{1/8}|c_k(f)| < \infty$ and $\sum_{k\in\mathbb{Z}}|k|^2|c_k(f)| = \infty$. Lastly, one checks directly that $f_5 \in C^2_{\text{per}}\backslash C^3_{\text{per}}$, then, for instance, $\sum_{k\in\mathbb{Z}}|k|^{1/8}|c_k(f_5)| < \infty$ and $\sum_{k\in\mathbb{Z}}|k|^3|c_k(f_5)| = \infty$.

Solution of 7.3:

1. In order to show the estimate, we first make the simple observation that

$$\int_0^\pi |D_n(x)|dx = \int_0^\pi \left|\frac{\sin((n+1/2)x)}{\sin(x/2)}\right|dx \geq \int_0^\pi 2\left|\frac{\sin((n+1/2)x)}{x}\right|dx,$$

since $|\sin(t)| \leq |t|$ for any $t \in \mathbb{R}$. Next we change variables setting $y(x) = \left(n + \frac{1}{2}\right) \cdot x$ to obtain

$$\begin{aligned}\int_0^\pi 2\left|\frac{\sin((n+1/2)x)}{x}\right|dx &= \int_0^\pi 2\underbrace{\left(n+\frac{1}{2}\right)}_{=y'(x)}\left|\frac{\sin(y(x))}{y(x)}\right|dx \\ &= 2\int_0^{(n+1/2)\pi}|\sin(y)|\frac{dy}{y} \\ &> 2\int_0^{n\pi}|\sin(y)|\frac{dy}{y}.\end{aligned}$$

By diving the domain of the latter integral into intervals of length π and plugging the result into the first estimate above, we directly obtain

$$\int_0^\pi |D_n(x)|dx 2 \cdot \sum_{j=0}^{n-1}\int_{j\pi}^{(j+1)\pi}|\sin(y)|\frac{dy}{y}.$$

2. Next we estimate the integral over the subintervals. First note that for any $j \in \mathbb{N}$ we obtain, changing variables $z := y - j\pi$,

$$\int_{j\pi}^{(j+1)\pi}|\sin(y)|\frac{dy}{y} = \int_0^\pi |\sin(z+j\pi)|\frac{dz}{z+j\pi} = \int_0^\pi |\sin(z)|\frac{dz}{z+j\pi},$$

where we used that $|\sin|$ is π-periodic. We further note that

$$\int_0^\pi |\sin(z)|\frac{dz}{y+j\pi} \geq \int_0^\pi |\sin(z)|\frac{dz}{(1+j)\pi} = \frac{1}{(1+j)\pi}\int_0^\pi |\sin(z)|dz.$$

Now, setting
$$c := \frac{1}{\pi}\int_0^\pi |\sin(y)|dy = \frac{1}{\pi}\left[-\cos(y)\right]_0^\pi = \frac{2}{\pi}$$
we obtain
$$\int_{j\pi}^{(j+1)\pi} |\sin(y)|\frac{dy}{y} \geq \frac{c}{j+1}$$
as a reformulation of our estimates above.

3. Using part (1) and (2) we obtain
$$\begin{aligned}\|D_n\|_{L^1(0,\pi)} = \int_0^\pi |D_n(x)|dx &> 2\cdot\sum_{j=0}^{n-1}\int_{j\pi}^{(j+1)\pi}|\sin(y)|\frac{dy}{y} \\ &\geq 2\cdot\sum_{j=0}^{n-1}\frac{1}{j+1}c = 2c\sum_{j=1}^{n}\frac{1}{j} \\ &= 2c\cdot H_n \asymp \log n,\end{aligned}$$
where we use $2c > 0$. Thus,
$$\|D_n\|_{L^1(0,\pi)} \geq O(\log n)$$
as $n \to \infty$.

Solution of 7.4:

Let $f, g \in L^2([-\pi,\pi];\mathbb{C})$. First, we point out that the Fourier coefficients of fg are well-defined by Cauchy–Schwarz, since
$$c_k(fg) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)e^{-ikx}dx,$$
and we have
$$\begin{aligned}|c_k(fg)| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)g(x)|\,dx \\ &\overset{\text{C.S}}{\leq} \frac{1}{2\pi}\left(\int_{-\pi}^{\pi}|f(x)|^2\,dx\right)^{1/2}\left(\int_{-\pi}^{\pi}|g(x)|^2\,dx\right)^{1/2} \\ &= \frac{1}{2\pi}||f||_{L^2}\,||g||_{L^2} < \infty.\end{aligned}$$
Furthermore, since f, g are L^2 functions, we know that as $N \to \infty$ we have
$$S_N(f) \to f \text{ and } S_N(g) \to g \text{ in } L^2.$$
Now, using the hints we obtain, for each $k \in \mathbb{Z}$
$$c_k(fg) = \lim_N c_k(S_N(f)S_N(g)), \tag{1}$$
since $S_N(f)S_N(g) \to fg$ in L^1.

Then we find for any fixed $k \in \mathbb{Z}$ and $N > |k|$

$$\begin{aligned}
c_k(S_N(f)S_N(g)) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{|j|\le N} c_j(f)e^{ijx}\right)\left(\sum_{|\ell|\le N} c_n(g)e^{i\ell x}\right)e^{-ikx}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{(j,\ell)\in[-N,N]^2} c_j(f)c_\ell(g)e^{i(j+\ell-k)x}dx \\
&= \frac{1}{2\pi}\sum_{(j,\ell)\in[-N,N]^2} c_j(f)c_\ell(g)\underbrace{\int_{-\pi}^{\pi} e^{i(j+\ell-k)x}dx}_{=\delta_{j+\ell-k}} \\
&= \sum_{\{(j,\ell)\in\mathbb{Z}^2 : j+\ell=k, |j|\le N, |\ell|\le N\}} c_j(f)c_\ell(g) \\
&= \sum_{|m|\vee|k-m|\le N} c_{k-m}(f)c_m(g),
\end{aligned}$$

so, combining this computation with (1), we can conclude provided we show that

$$\lim_{N\to\infty}\sum_{|m|\vee|k-m|\le N} c_{k-m}(f)c_m(g) = \sum_{m\in\mathbb{Z}} c_{k-m}(f)c_m(g).$$

This is a consequence of the dominated convergence Theorem in the measure space $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \#)$ since it is equivalent to show that the integral (in this measure space) of

$$\phi_N(m) := \begin{cases} c_{k-m}(f)c_m(g) & \text{if } |m|\vee|k-m|\le N, \\ 0 & \text{otherwise} \end{cases}$$

converge to the integral of $\phi(m) := c_{k-m}(f)c_m(g)$. But then clearly

$$|\phi_N(m)| \le |\phi(m)| \text{ for all } m \in \mathbb{Z},$$

and $\phi \in L^1(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \#)$ since

$$\sum_{j\in\mathbb{Z}} |c_j(f)\, c_{k-j}(g)| \le \left(\sum_{j\in\mathbb{Z}} |c_j(f)|^2\right)^{1/2}\left(\sum_{k\in\mathbb{Z}} |c_k(g)|^2\right)^{1/2} \overset{\text{Parseval}}{\le} \|f\|_2\|g\|_2 < \infty.$$

Which completes our proof, since we also proved that the series $\sum_{j\in\mathbb{Z}} |c_j(f)\, c_{k-j}(g)| < \infty$, i.e., it is absolutely convergent.