The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

8.1. Closed answer questions.

1. If $f \in C^2(\mathbb{R})$ and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$, then necessarily $f|_{[-\pi,\pi]} \in C^2_{per}$? What about the viceversa?
2. For which values of $\alpha \in \mathbb{R}$ and $p \geq 1$ we have that $\{k^\alpha\} \in \ell^p(\mathbb{N} \setminus \{0\})$? **Hint**: Recall that $\sum_{k\geq 1} k^s < \infty \iff s < -1$.
3. Does it exists a continuous and periodic function f such that $c_k(f) \asymp |k|^{-1/3} \log |k|$ as $k \to \infty$?
4. Does it exist a function $f \in L^1(-\pi,\pi)$ such that $c_k(f) \not\to 0$ as $|k| \to \infty$? **Hint**: Riemann-Lebesgue lemma.
5. Give an example of a C^∞ and 2π-periodic function which is not a trigonometric polynomial. Can you make it analytic? **Hint**: try with $\exp(\exp(ix))$.

8.2. Formal solutions of PDEs. For the following PDEs of evolution type try to find the most general solution of the form $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{-ikx}$, without worrying about convergence issues. Of course the functions $\{u_k(t)\}_{k\in\mathbb{Z}}$ might depend on the Fourier coefficients of $u(0,\cdot)$ (and sometimes also of $\partial_t u(0,\cdot)$)

1. $\partial_t u = \cos(t)\partial_{xx} u$
2. $\partial_{tt} u - \partial_{xx} u = 0$
3. $\partial_t u = \frac{1}{1+t^2} u + \partial_{xx} u$
4. (BONUS) $\partial_t u = \partial_{xx} u + 1$

For each of these cases write down an example of solution which is not a constant. **Remark**: you get the bonus point if you write both the most general solution in point 4 and an example.

8.3. Free Schrödinger equation in a ring. Consider the evolution problem with periodic boundary conditions:

$$\begin{cases} i\partial_t u + \partial_{xx} u = 0 & \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(t,x) = u(t,x+2\pi) & \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = f(x) & \text{for some given } f \in C^\infty(\mathbb{R}), 2\pi\text{-periodic.} \end{cases}$$

1. Explain why solutions cannot be purely real-valued, unless they are constant.

2. Explain why, for each fixed large N, we have $\sup_k |k|^N |c_k(f)| < \infty$.

3. Write the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, where the $\{u_k(t)\}$ depend on the Fourier coefficients of f.

4. Show that the formal solution is in fact a true solution and is C^∞ in both variables. **Hint**: you need to show that the coefficients $\{c_k(\partial_t^m \partial_x^n u(t,\cdot))\}$ are summable. This follows from the decay of the $\{c_k(f)\}$. It might be convenient to show uniform (with respect to N) bounds on the mixed derivatives of
$$u_N(t,x) := \sum_{|k|\le N} u_k(t)e^{-ikx}.$$

5. Show that we found the only possible solution: if v is a solution of the problem which is C^2_{per} in space and C^1 in time, then $u = v$. **Hint**: argue exactly as in the proof of uniqueness for the heat equation.

6. Write explicitly u in the case $f = 2\cos(3x)$.

7. Does this equation enjoy the "smoothing effect" of the heat equation? **Hint**: observe that the size of u_k and the size of $c_k(f)$ are comparable: do we expect regularisation?