The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with $(*)$ can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

8.1. Closed answer questions.

1. If $f \in C^2(\mathbb{R})$ and $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$, then necessarily $f|_{[-\pi,\pi]} \in C^2_{per}$? What about the viceversa?
2. For which values of $\alpha \in \mathbb{R}$ and $p \geq 1$ we have that $\{k^\alpha\} \in \ell^p(\mathbb{N} \setminus \{0\})$? **Hint**: Recall that $\sum_{k\geq 1} k^s < \infty \iff s < -1$.
3. Does it exists a continuous and periodic function f such that $c_k(f) \asymp |k|^{-1/3} \log |k|$ as $k \to \infty$?
4. Does it exist a function $f \in L^1(-\pi,\pi)$ such that $c_k(f) \not\to 0$ as $|k| \to \infty$? **Hint**: Riemann-Lebesgue lemma.
5. Give an example of a C^∞ and 2π-periodic function which is not a trigonometric polynomial. Can you make it analytic? **Hint**: try with $\exp(\exp(ix))$.

8.2. Formal solutions of PDEs. For the following PDEs of evolution type try to find the most general solution of the form $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t) e^{-ikx}$, without worrying about convergence issues. Of course the functions $\{u_k(t)\}_{k\in\mathbb{Z}}$ might depend on the Fourier coefficients of $u(0,\cdot)$ (and sometimes also of $\partial_t u(0,\cdot)$)

1. $\partial_t u = \cos(t)\partial_{xx} u$
2. $\partial_{tt} u - \partial_{xx} u = 0$
3. $\partial_t u = \frac{1}{1+t^2} u + \partial_{xx} u$
4. (BONUS) $\partial_t u = \partial_{xx} u + 1$

For each of these cases write down an example of solution which is not a constant. **Remark**: you get the bonus point if you write both the most general solution in point 4 and an example.

8.3. Free Schrödinger equation in a ring. Consider the evolution problem with periodic boundary conditions:

$$\begin{cases} i\partial_t u + \partial_{xx} u = 0 & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(t,x) = u(t,x+2\pi) & \text{for all } (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = f(x) & \text{for some given } f \in C^\infty(\mathbb{R}), 2\pi\text{-periodic}. \end{cases}$$

1. Explain why solutions cannot be purely real-valued, unless they are constant.

2. Explain why, for each fixed large N, we have $\sup_k |k|^N |c_k(f)| < \infty$.
3. Write the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, where the $\{u_k(t)\}$ depend on the Fourier coefficients of f.
4. Show that the formal solution is in fact a true solution and is C^∞ in both variables. **Hint**: you need to show that the coefficients $\{c_k(\partial_t^m \partial_x^n u(t,\cdot))\}$ are summable. This follows from the decay of the $\{c_k(f)\}$. It might be convenient to show uniform (with respect to N) bounds on the mixed derivatives of
$$u_N(t,x) := \sum_{|k|\le N} u_k(t)e^{-ikx}.$$
5. Show that we found the only possible solution: if v is a solution of the problem which is C^2_{per} in space and C^1 in time, then $u = v$. **Hint**: argue exactly as in the proof of uniqueness for the heat equation.
6. Write explicitly u in the case $f = 2\cos(3x)$.
7. Does this equation enjoy the "smoothing effect" of the heat equation? **Hint**: observe that the size of u_k and the size of $c_k(f)$ are comparable: do we expect regularisation?

8. Solutions

Solution of 8.1:

1. It's true. Recall that the functions in $C^2_{per}([-\pi,\pi])$ are the functions of class C^2 satisfying the periodicity conditions
$$f^{(k)}(\pi) = f^{(k)}(-\pi) \quad k = 0,1,2 \tag{1}$$
If f is a 2π-periodic C^2 function, then
$$f'(x) = \frac{d}{dx}\bigg|_x f(\cdot) = \frac{d}{dx}\bigg|_x f(\cdot + 2\pi) = f'(x+2\pi)$$
and, by iteration, the same holds for f'', thus $f \in C^2_{per}$. The vice-versa is also true, the only thing to check is that the periodic extension of a function $f \in C^2_{per}([-\pi,\pi])$ is differentiable in the "junction points" $\{(2k+1)\pi\}_{k\in\mathbb{Z}}$, but this is exactly given by the boundary conditions (1).
2. Consider first $p = \infty$, then we obtain $\alpha \le 0$ since $\{k^\alpha\}_{k\in\mathbb{N}}$ is bounded only in that case. Let $p \ge 1$ now be finite. We have
$$\{k^\alpha\}_{k\in\mathbb{N}} \in \ell^p(\mathbb{N}) \iff \sum_{k=1}^\infty (k^\alpha)^p < \infty.$$
Note that
$$\sum_{k=1}^\infty (k^\alpha)^p = \sum_{k=1}^\infty k^{\alpha\cdot p} < \infty \iff \alpha\cdot p < -1,$$
which can easily be seen using the integral test for the convergence of a series (a.k.a the Cauchy-Maclaurin test). Thus, $\{k^\alpha\}_{k\in\mathbb{N}} \in \ell^p(\mathbb{N})$ if and only if $\alpha < -1/p$.

3. If $f \in C_{per}$ then in particular $f \in L^2$ so by Parseval's theorem

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 < \infty. \tag{2}$$

Now we have

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-2/3}(\log|k|)^2 \geq 2\sum_{k=1}^{\infty}|k|^{-2/3}(\log|k|)^2 \geq 2\log 2 \underbrace{\sum_{k=2}^{\infty}|k|^{-2/3}}_{=\infty},$$

where we again use that $\sum_{k\geq 1} k^s < \infty$ if and only if $s < -1$ by the Cauchy-Maclaurin test for the convergence of a series. Thus, if $c_k(f) \asymp |k|^{-1/3}\log|k|$ we have $\sum_{k\in\mathbb{Z}}|c_k(f)|^2 = \infty$ which contradicts (2).

4. First note that

$$\int_{-\pi}^{\pi} e^{-ix}dx = \left[-i \cdot e^{-ix}\right]_{-\pi}^{\pi} = 0.$$

Let $g \in L^1(-\pi,\pi)$. Then by the Riemann-Lebesgue lemma we get that

$$\lim_{k\to\infty} \underbrace{\int_{-\pi}^{\pi} g(x)\cdot e^{-ikx}dx}_{=c_k(g)} = 0 \cdot \int_{-\pi}^{\pi} g(x)dx = 0,$$

which implies that $c_k(g) \to 0$ for any L^1-function g on $(-\pi,\pi)$.

5. A function is a trigonometric polynomial if it is a finite sum of functions in the Hilbert basis $\{e^{ikx}\}_{k\in\mathbb{Z}}$. Assume $f \in L^1(-\pi,\pi)$ is a trigonometric polynomial. Given the uniqueness of the coefficients this implies that there is some $N \in \mathbb{N}$ such that $c_k(f) = 0$ for all $k > N$.

Let now $f(x) = \exp(\exp(ix))$, where $\exp\colon \mathbb{C} \to \mathbb{C}$ is the complex exponential, clearly $f \in C^{\infty}_{per}$. Note that[1]

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}(\exp(ix))^k = \sum_{k=0}^{\infty} \frac{1}{k!}\exp(ixk),$$

where the convergence of the series is uniform for $x \in [-\pi,\pi]$ since, for x varying in that range, $|\exp(ikx)| \leq 1$.

Now, since $\{\exp(ikx)\}_{k\in\mathbb{Z}}$ is an orthonormal system (and even a Hilbert basis) the coefficients with respect to it are unique. By the above we thus obtain that $c_k(f) = \frac{1}{k!} > 0$ for $k \in \mathbb{N}$. This implies that f cannot be a trigonometric polynomial. Lastly, note that f is analytic since it is the composition of analytic functions.

Solution of 8.2:

[1] We are using that exp is an entire function and so $\exp(z) = \sum_{k\in\mathbb{N}} z^k/k!$ for all $z \in \mathbb{C}$, so we can substitute $z = \exp(ikx)$.

1. We write u as $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, notice that
$$\partial_t u = \sum_{k\in\mathbb{Z}} u_k'(t)e^{ikx} \text{ and } \cos(t)\partial_{xx}u = \sum_{k\in\mathbb{Z}} -k^2\cos(t)u_k(t)e^{ikx}.$$
Hence we get the differential equations $u_k'(t) = -k^2\cos(t)u_k(t)$, for all $k \in \mathbb{Z}$. Then, $u_k(t) = ce^{-k^2\sin(t)}$, where $c = u_k(0)$. So,
$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k(0)e^{-k^2\sin(t)+ikx}.$$
Example: $u(t,x) = e^{-\sin(t)+ix}$.

2. We write u as $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, notice that
$$\partial_{tt}u = \sum_{k\in\mathbb{Z}} u_k''(t)e^{ikx} \text{ and } \partial_{xx}u = \sum_{k\in\mathbb{Z}} -k^2u_k(t)e^{ikx},$$
hence we find the differential equations $u_k''(t) = -k^2u_k(t)$, for all $k \in \mathbb{Z}$. For $k = 0$ this gives $u_0(t) = u_0(0) + u_0'(0)t$, while for $k \neq 0$ we have $u_k(t) = c_1e^{ikt} + c_2e^{-ikt}$, where $c_1 = \frac{1}{2}(u_k(0) - \frac{i}{k}u_k'(0))$ and $c_2 = \frac{1}{2}(u_k(0) + \frac{i}{k}u_k'(0))$. So we find
$$u(t,x) = u_0(0)+u_0'(0)t+\sum_{k\in\mathbb{Z}\setminus\{0\}} \tfrac{1}{2}(u_k(0)-\tfrac{i}{k}u_k'(0))e^{ik(t-x)}+\tfrac{1}{2}(u_k(0)+\tfrac{i}{k}u_k'(0))e^{-ik(t+x)}.$$
Example: $u(t,x) = \sin(t-x)$.

3. We write u as $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, notice that
$$\partial_t u = \sum_{k\in\mathbb{Z}} u_k'(t)e^{ikx}, \quad \tfrac{1}{1+t^2}u = \sum_{k\in\mathbb{Z}} \tfrac{1}{1+t^2}u_k(t)e^{ikx} \text{ and } \partial_{xx}u = \sum_{k\in\mathbb{Z}} -k^2u_k(t)e^{ikx}.$$
Hence we get the differential equations $u_k'(t) = \frac{1}{1+t^2}u_k(t) - k^2u_k(t)$, for all $k \in \mathbb{Z}$. Then we find $u_k(t) = ce^{\arctan(t)-k^2t}$, where $c = u_k(0)$. Hence,
$$u(t,x) = \sum_{k\in\mathbb{Z}} u_k(0)e^{\arctan(t)-k^2t+ikx}.$$
Example: $u(t,x) = e^{\arctan(t)-t+ix}$.

4. As usual, we write $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$ and observe that u formally satisfies
$$\sum_{k\in Z} u_k'(t)e^{ikx} = \sum_{k\in Z}(-k^2)u_k(t)e^{ikx} + 1.$$
For $k = 0$, we have $u_0' = 1$ and thus $u_0(t) = u_0(0) + t$. For $k \neq 0$, we have
$$u_k(t) = u_k(0)e^{-k^2t}.$$
The most general solution is
$$u(t,x) = u_0(0) + t + \sum_{k\in\mathbb{Z}\setminus\{0\}} u_k(0)e^{-k^2t}e^{ikx}.$$
As an example we can choose $u(t,x) = t$.

Solution of 8.3:

1. Suppose we have a real-valued solution u which is not identically constant. Then all derivatives of u also have to be real valued. This implies that, at any given point (t, x)
$$i\mathbb{R} \ni i\partial_t u(x,t) = -\partial_{xx} u(x,t) \in \mathbb{R} \implies \partial_t u = \partial_{xx} u = 0.$$
This implies that, at any point (t, x), u needs to be constant in time and affine in space, i.e. $u(t,x) = a + bx$. The periodic boundary conditions force $b = 0$, so u is constant.

2. Note that the statement in Exercise 8.1.1 is still true if we replace 2 with an arbitrary N, that is, $f \in C^N_{per}$ for all $N \in \mathbb{N}$.

   Then Theorem 2.22 (ii) implies that $\sum_k |k|^\alpha |c_k(f)| < \infty$ for all $\alpha \geq 0$. Thus $|k|^\alpha |c_k(f)| \to 0$ as $k \to \infty$, in particular $\sup_k |k|^N |c_k(f)| < \infty$ for each individual N.

3. If we make the Ansatz $u(t,x) = \sum_k u_k(t) e^{ikx}$ and derive u formally, we get (similar to the case of the heat equation):
$$i\partial_t u(t,x) = \sum_{k\in\mathbb{Z}} i u_k'(t) e^{ikx} \text{ and } \partial_{xx} u(t,x) = \sum_{k\in\mathbb{Z}} -k^2 u_k(t) e^{ikx}.$$
Imposing $i\partial_t u + \partial_{xx} u = 0$, the coefficient functions $u_k(t)$ have to solve the following ODE:
$$\begin{cases} u_k'(t) = -ik^2 u_k(t) \\ u_k(0) = c_k(f), \end{cases}$$
which is solved by $u_k(t) = c_k(f) e^{-ik^2 t}$. So we can write our general formal solution to the Schrödinger equation as follows:
$$u(t,x) = \sum_{k\in\mathbb{Z}} c_k(f) e^{-ik^2 t} e^{ikx}. \tag{3}$$

4. We begin first by showing smoothness. Define
$$w_k(t,x) = c_k(f) e^{-ik^2 t} e^{ikx}.$$
We show that for $m, n \in \mathbb{N}$, the sums
$$\sum_{k\in\mathbb{Z}} \partial_t^m \partial_x^n w_k(t,x)$$
converge absolutely and uniformly on $\mathbb{R} \times \mathbb{R}$. Note that
$$|\partial_t^m \partial_x^n w_k(t,x)| = |(-ik^2)^m (ik)^n c_k(f) e^{-ik^2 t} e^{ikx}| = |k|^{2m+n} |c_k(f)|,$$
for all $t, x \in \mathbb{R}$. Set now $N = 2m + n + 2$. Thanks to item 2, we know that $\sup_k |k|^N |c_k(f)| < \infty$, i.e. there is some constant $C_N \geq 0$ such that $|k|^N |c_k(f)| \leq C_N$ for all $k \in \mathbb{Z}$.

It follows that

$$\begin{aligned}\sup_{(t,x)\in\mathbb{R}\times\mathbb{R}}\sum_{|k|\geq M}|\partial_t^m\partial_x^n w_k(t,x)| &= \sum_{|k|\geq M}|k|^{2m+n}|c_k(f)|\\ &\leq \sum_{|k|\geq M} C_N|k|^{-N}|k|^{2m+n}\\ &= \sum_{|k|\geq M} C_N|k|^{-2}\underset{M\to\infty}{\longrightarrow} 0.\end{aligned}$$

Thus, the sums converge indeed absolutely and uniformly and the previously formally defined function u is actually well-defined. Moreover, the uniform convergence of the sum above implies that we can interchange summation and derivatives, that is:

$$\partial_t^m\partial_x^n u(t,x) = \partial_t^m\partial_x^n\sum_{k\in\mathbb{Z}} w_k(t,x) = \sum_{k\in\mathbb{Z}}\partial_t^m\partial_x^n w_k(t,x)$$

and hence $\partial_t^m\partial_x^n u(t,x)$ exists and is continuous. Since $m,n\in\mathbb{N}$ were arbitrary, this implies that $u\in C^\infty$.

It follows now directly from the remark above about interchanging derivatives and summation that u actually solves the Schrödinger equation:

$$\begin{aligned}i\partial_t u(t,x) = i\partial_t\sum_{k\in\mathbb{Z}} c_k(f)e^{-ik^2t}e^{ikx} &= \sum_{k\in\mathbb{Z}} i\partial_t c_k(f)e^{-ik^2t}e^{ikx}\\ &= \sum_{k\in\mathbb{Z}} k^2c_k(f)e^{-ik^2t}e^{ikx} = -\sum_{k\in\mathbb{Z}}\partial_{xx}c_k(f)e^{-ik^2t}e^{ikx}\\ &= -\partial_{xx}\sum_{k\in\mathbb{Z}} c_k(f)e^{-ik^2t}e^{ikx} = -\partial_{xx}u(t,x).\end{aligned}$$

5. Let v be as described another solution with the same initial data and write

$$v(t,x) = \sum_{k\in\mathbb{Z}} d_k(t)e^{ikx},$$

where $d_k(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)e^{-ikx}\,\mathrm{d}x = c_k(v(t,\cdot))$.

As in the proof for the heat equation, we first show that $d_k(t)\in C^1(\mathbb{R})$ for all k. For this, let $t\in\mathbb{R}$. We want to show that $\lim_{s\to t} d_k(s) = d_k(t)$. Since v is continuous, it is bounded on the (compact) rectangle $[t-1,t+1]\times[-\pi,\pi]$ by some constant K_t. Thus for s close enough, we can dominate $v(s,x)e^{-ikx}$ by the constant function K_t. Now, by dominated convergence and continuity of v we get:

$$\lim_{s\to t} d_k(s) = \lim_{s\to t}\frac{1}{2\pi}\int_{-\pi}^{\pi} v(s,x)e^{-ikx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\lim_{s\to t} v(s,x)}_{=v(t,x)}e^{-ikx}\,\mathrm{d}x = d_k(t).$$

Since by assumption also $\partial_t v$ is continuous, we can interchange integral and differentiation

$$d_k'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)e^{-ikx}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \partial_t v(t,x)e^{-ikx}\,\mathrm{d}x.$$

Continuity of $d_k'(t)$ is shown exactly like above, just with v replaced by $\partial_t v$. Thus $d_k \in C^1(\mathbb{R})$.

Next, define $F_k(t) = e^{ik^2t}d_k(t)$. F_k is differentiable and

$$\begin{aligned}
F_k'(t) &= ik^2e^{ik^2t}d_k(t) + e^{ik^2t}d_k'(t) \\
&= ik^2e^{ik^2t}d_k(t) + e^{ik^2t}c_k(\partial_t v(t,\cdot)) \\
&= ik^2e^{ik^2t}d_k(t) + ie^{ik^2t}c_k(\partial_{xx} v(t,\cdot)) \\
&= ik^2e^{ik^2t}d_k(t) - ik^2e^{ik^2t}\underbrace{c_k(v(t,\cdot))}_{d_k(t)} = 0,
\end{aligned}$$

for all $t \in \mathbb{R}$. So F_k is a constant function with $F_k(t) \equiv c_k(f)$. Hence for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}$, it holds that $d_k(t) = c_k(f)e^{-ik^2t}$.

We can conclude that

$$v(t,x) = \sum_k d_k(t)e^{ikx} = \sum_k c_k(f)e^{-ik^2t}e^{ikx} = u(t,x),$$

where the equality holds in L^2, so almost everywhere. But both u and v are assumed to be continuous, so equality holds actually everywhere. This shows that u has to be the unique solution for the Schrödinger equation.

6. We can write $f(x)$ as $e^{3ix} + e^{-3ix}$. Inserting this into (3), we get the solution $u(t,x) = e^{-9it}\left(e^{3ix} + e^{-3ix}\right) = 2e^{-9it}\cos(3x)$.

7. Recall that for the heat equation, the solution is smooth for positive times, even if the initial data is not smooth (for instance if it is only C^1_{per}). If v is a solution to the heat equation with initial data f, then $v(t,x) = \sum_k c_k(f)e^{-k^2t}e^{ikx}$. The factor e^{-k^2t} is what gives v its regularity, as it dominates – for $t > 0$ – any polynomial in k. If we compare this to a solution u of the Schrödinger equation with the same initial data, i.e. $u(t,x) = \sum_k c_k(f)e^{-ik^2t}e^{ikx}$, we see that we need fast decay of the $c_k(f)$ in order to have regularity, since the factor e^{-ik^2t} has modulus 1, hence it does not contribute to convergence of the seires. (Compare the calculations in point 4, where the fast decay of the $c_k(f)$ was crucial, with the calculations you did in class in the Proof of Theorem 2.33 (iii)).

   So in general, the regularity of solutions to the Schrödinger equation is dependent on the regularity of the boundary data, contrary to the heat equation.