The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

9.1. Closed answer questions.

1. If g_k are continuous and compactly supported functions in $\mathbb{R}^d$ such that $g_k \to g$ uniformly, is it true that g is necessarily continuous? Vanishes as $|x| \to \infty$? Has compact support?
2. Let $\phi \in L^1(\mathbb{R}^d)$ and consider $\phi_t(x) := \phi(x)\mathbf{1}_{\{|\phi(x)|\geq t\}}$, for $t > 0$. Is it true that

$$\sup_{\xi\in\mathbb{R}^d} |\mathcal{F}(\phi_t)(\xi)| \to 0 \text{ as } t \to \infty?$$

   Hint: $\|\phi_t\|_{L^1} \to 0$ as $t \to \infty$... (why?).
3. Compute the Fourier transform of the indicator function of the interval $\mathbf{1}_{[-1,1]}(x)$, for $x \in \mathbb{R}$.
4. Given $f \in L^1(\mathbb{R}^d)$ define $f * f$ and explain why $(f * f)(0)$ is not necessarily a well-defined number (an example suffices).

9.2. Properties of the Fourier transform. (BONUS) Determine which of the following statements is true. Select all that apply.

1. If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^1(\mathbb{R}^d)$.
2. If f is compactly supported, then $\hat{f} \in L^1(\mathbb{R}^d)$.
3. If $\hat{f}(\xi_1, \xi_2) = \frac{\sin(\xi_2)}{1+i\xi_1^2}$ then $f \in L^1(\mathbb{R}^2)$.
4. If f is compactly supported and bounded, then $\hat{f} \in \mathcal{C}_0(\mathbb{R}^d)$[1].

9.3. Heat equation for rough initial data. You are given $f \in L^2(-\pi, \pi)$, and consider the associated heat equation solution defined by

$$u(t, x) := \sum_{k\in\mathbb{Z}} c_k(f)e^{ikx-k^2t}, \text{ for all } x \in \mathbb{R}, t > 0.$$

1. Show that $u \in C^\infty((0, \infty) \times \mathbb{R})$ and solves the heat equation

$$\partial_t u(t, x) = \partial_{xx} u(t, x), \qquad \text{for all } x \in \mathbb{R}, t > 0. \tag{1}$$

   Hint: Start from Parseval's inequality and argue as in the heat equation proof.

[1] That is, continuous functions which vanish at infinity

2. Show that u assumes the initial datum f in the following L^2 sense
$$\lim_{t\downarrow 0} \|u(t,\cdot) - f\|_{L^2(-\pi,\pi)} = 0. \tag{2}$$
3. Consider a function $v(t,x)$ defined in $(0,\infty)\times\mathbb{R}$ which is of class C^2 in space and 2π-periodic, and of class C^1 in time. If v satisfies equations (1) and (2), then $v = u$.

9.4. The wave equation.

Consider the evolution problem with periodic boundary conditions:

$$\begin{cases} \partial_{tt}u - \partial_{xx}u = \lambda u & \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R}, \text{ where } \lambda\le 0 \text{ is a given constant,}\\ u(t,x) = u(t,x+2\pi) & \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R},\\ u(0,x) = f(x) & \text{for some given } f\in C^\infty(\mathbb{R}), 2\pi\text{-periodic,}\\ \partial_t u(0,x) = g(x) & \text{for some given } g\in C^\infty(\mathbb{R}), 2\pi\text{-periodic.}\end{cases}$$

1. Write the most general formal solution $u(t,x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx}$, where the $\{u_k(t)\}$ depend on λ and the Fourier coefficients of f and g. **Hint**: Recall that λ has a sign, you will get the equation for an harmonic oscillator.
2. Show that the formal solution is in fact a true solution and is C^∞ in both variables.
3. Show that, if we just want our solution u to be $C^2(\mathbb{R}\times\mathbb{R})$, the assumptions on f and g can be relaxed to:
$$\sum_{k\in\mathbb{Z}} |k|^2|c_k(f)| + |k||c_k(g)| < +\infty.$$
4. Assume that $\lambda = 0$. Show that for each pair $\phi,\psi \in C^2_{per}$ the function $(x,t)\mapsto \phi(x-t)+\psi(x+t)$ solves the wave equation, explain why this is compatible with what you found in the previous points.
5. ($\star$) Does the wave equation have the "smoothing effect" for positive times?