The exercises below are listed by increasing difficulty, starting from warm-up questions that serve to get acquainted with the topics, up to exam-like questions. Questions marked with (∗) can be challenging and are more difficult than the average exam question. You are encouraged to try and solve them by working in groups if necessary.

The question marked with BONUS is a multiple-choice question that can contribute to extra points in the final exam; refer to the webpage for more information.

9.1. Closed answer questions.

1. If g_k are continuous and compactly supported functions in $\mathbb{R}^d$ such that $g_k \to g$ uniformly, is it true that g is necessarily continuous? Vanishes as $|x| \to \infty$? Has compact support?
2. Let $\phi \in L^1(\mathbb{R}^d)$ and consider $\phi_t(x) := \phi(x)\mathbf{1}_{\{|\phi(x)|\geq t\}}$, for $t > 0$. Is it true that

$$\sup_{\xi\in\mathbb{R}^d} |\mathcal{F}(\phi_t)(\xi)| \to 0 \text{ as } t \to \infty?$$

   Hint: $\|\phi_t\|_{L^1} \to 0$ as $t \to \infty$. . . (why?).
3. Compute the Fourier transform of the indicator function of the interval $\mathbf{1}_{[-1,1]}(x)$, for $x \in \mathbb{R}$.
4. Given $f \in L^1(\mathbb{R}^d)$ define $f * f$ and explain why $(f * f)(0)$ is not necessarily a well-defined number (an example suffices).

9.2. Properties of the Fourier transform. (BONUS) Determine which of the following statements is true. Select all that apply.

1. If $f \in L^1(\mathbb{R}^d)$, then $\hat{f} \in L^1(\mathbb{R}^d)$.
2. If f is compactly supported, then $\hat{f} \in L^1(\mathbb{R}^d)$.
3. If $\hat{f}(\xi_1, \xi_2) = \frac{\sin(\xi_2)}{1+i\xi_1^2}$ then $f \in L^1(\mathbb{R}^2)$.
4. If f is compactly supported and bounded, then $\hat{f} \in \mathcal{C}_0(\mathbb{R}^d)$[1].

9.3. Heat equation for rough initial data. You are given $f \in L^2(-\pi, \pi)$, and consider the associated heat equation solution defined by

$$u(t, x) := \sum_{k\in\mathbb{Z}} c_k(f)e^{ikx-k^2t}, \text{ for all } x \in \mathbb{R}, t > 0.$$

1. Show that $u \in C^\infty((0, \infty) \times \mathbb{R})$ and solves the heat equation

$$\partial_t u(t, x) = \partial_{xx}u(t, x), \qquad \text{for all } x \in \mathbb{R}, t > 0. \tag{1}$$

   Hint: Start from Parseval's inequality and argue as in the heat equation proof.

[1] That is, continuous functions which vanish at infinity

2. Show that u assumes the initial datum f in the following L^2 sense

$$\lim_{t \downarrow 0} \|u(t, \cdot) - f\|_{L^2(-\pi,\pi)} = 0. \tag{2}$$

3. Consider a function $v(t, x)$ defined in $(0, \infty) \times \mathbb{R}$ which is of class C^2 in space and 2π-periodic, and of class C^1 in time. If v satisfies equations (1) and (2), then $v = u$.

9.4. The wave equation.

Consider the evolution problem with periodic boundary conditions:

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = \lambda u & \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \text{ where } \lambda \leq 0 \text{ is a given constant,} \\ u(t,x) = u(t, x + 2\pi) & \text{for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = f(x) & \text{for some given } f \in C^\infty(\mathbb{R}), 2\pi\text{-periodic,} \\ \partial_t u(0,x) = g(x) & \text{for some given } g \in C^\infty(\mathbb{R}), 2\pi\text{-periodic.} \end{cases}$$

1. Write the most general formal solution $u(t, x) = \sum_{k \in \mathbb{Z}} u_k(t) e^{ikx}$, where the $\{u_k(t)\}$ depend on λ and the Fourier coefficients of f and g. **Hint**: Recall that λ has a sign, you will get the equation for an harmonic oscillator.
2. Show that the formal solution is in fact a true solution and is C^∞ in both variables.
3. Show that, if we just want our solution u to be $C^2(\mathbb{R} \times \mathbb{R})$, the assumptions on f and g can be relaxed to:

$$\sum_{k \in \mathbb{Z}} |k|^2 |c_k(f)| + |k||c_k(g)| < +\infty.$$

4. Assume that $\lambda = 0$. Show that for each pair $\phi, \psi \in C^2_{per}$ the function $(x, t) \mapsto \phi(x - t) + \psi(x + t)$ solves the wave equation, explain why this is compatible with what you found in the previous points.
5. ($\star$) Does the wave equation have the "smoothing effect" for positive times?

9. Solutions

Solution of 9.1:

1. Recall that $(C(\mathbb{R}^d), \|\cdot\|_{L^\infty})$ is a Banach space, hence g is continuous. We claim that $g(x) \to 0$ as $|x| \to \infty$. Indeed, for any $\epsilon > 0$ we can find $k \in \mathbb{N}$ such that $\|g_k - g\|_{L^\infty} < \epsilon$. Then, there is $R = R_k > 0$ large enough such that $\operatorname{supp}(g_k) \subset B_R(0)$. Now for all $x \in \mathbb{R}^d$ with $|x| \geq R$ we have

$$|g(x)| = |g(x) - g_k(x)| \leq \|g - g_k\|_{L^\infty} < \epsilon.$$

Thus $|g(x)| \to 0$ as $|x| \to \infty$.

Next, we show that with a counterexamople that g does not necessarily have compact support in $\mathbb{R}^d$. Let $v_k : [0, \infty) \to \mathbb{R}$,

$$v_k(r) = \begin{cases} 1, & \text{if } r \leq k, \\ -r + k + 1, & \text{if } k \leq r \leq k+1, \\ 0, & \text{if } r \geq k+1. \end{cases}$$

Let $g_k(x) = v_k(|x|)e^{-|x|^2}$ and $g(x) = e^{-|x|^2}$. Then g_k has compact support in $\mathbb{R}^d$ and $g_k \to g$, as $k \to \infty$ in L^∞. But g is not compactly supported in $\mathbb{R}^d$.

2. It is true. Indeed, by dominated convergence theorem we have $\phi_t \to 0$ in $L^1(\mathbb{R}^d)$, where we use as dominant ϕ itself. Remark that the pointwise limit is 0 since $|\{|\phi| = \infty\}| = 0$ (as it is L^1). By Theorem 3.3 we conclude

$$\|\widehat{\phi}_t\|_{L^\infty(\mathbb{R}^d)} \leq (2\pi)^{-d/2}\|\phi_t\|_{L^1(\mathbb{R}^d)} \to 0,$$

as $t \to \infty$.

3. For each $\xi \in \mathbb{R}$ compute

$$\begin{aligned} \widehat{\mathbf{1}_{[-1,1]}}(\xi) &= (2\pi)^{-1/2} \int_{-1}^{1} e^{-i\xi x}\, dx \\ &= (2\pi)^{-1/2} \left[\frac{1}{-i\xi} e^{-i\xi x}\right]_{-1}^{1} \\ &= (2\pi)^{-1/2} \frac{1}{i\xi}\left(e^{i\xi} - e^{-i\xi}\right) \\ &= \sqrt{\frac{2}{\pi}} \frac{\sin(\xi)}{\xi}. \end{aligned}$$

4. Using Theorem 3.7 (Young's Inequality) we obtain that $f * f \in L^1(\mathbb{R}^d)$ is well-defined and for a.e. $x \in \mathbb{R}^d$ we have the identity

$$(f * f)(x) = \int_{\mathbb{R}^d} f(x-y) f(y)\, dy.$$

There exist functions $f \in L^1(\mathbb{R}^d)$ such that the above identity does not hold for $x = 0$. Indeed, consider

$$f(x) = |x|^{-d/2}\mathbf{1}_{B_1(0)}(x).$$

Then $f \in L^1(\mathbb{R}^d)$, but

$$(f * f)(0) = \int_{\mathbb{R}^d} f(-y)f(y)\, dy = \int_{B_1(0)} |y|^{-d}\, dy = \infty.$$

Solution of 9.2:

1. False. Take for example $f = \chi_{[-1,1]} \in L^1(\mathbb{R})$, then

$$\hat{f}(\xi) = \sqrt{\frac{2}{\pi}}\frac{\sin \xi}{\xi} \notin L^1(\mathbb{R}).$$

   Indeed

$$\int_{\mathbb{R}} \left|\frac{\sin \xi}{\xi}\right| d\xi = 2\sum_{k=0}^{\infty}\int_{k\pi}^{(k+1)\pi} \frac{|\sin \xi|}{|\xi|} d\xi \geq 2\sum_{k=0}^{\infty}\int_{k\pi}^{(k+1)\pi} \frac{|\sin \xi|}{(k+1)\pi} d\xi = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{1+k} = \infty.$$

2. False. The same counterexample of point 1 still works.
3. False. If by contradiction f was $L^1(\mathbb{R}^2)$, then by Riemann–Lebesgue lemma we would have $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$. But

$$\hat{f}(0, \xi_2) = \sin(\xi_2) \not\to 0 \text{ as } |\xi_2| \to \infty.$$

4. True. If f is compactly supported and bounded, then $f \in L^1$ and the Fourier transform maps L^1 into $\mathcal{C}_0$.

Solution of 9.3:

1. As f is in $L^2(-\pi, \pi)$, we make use of Parseval's identity

$$2\pi \sum_{k\in\mathbb{Z}} |c_k(f)|^2 = ||f||_{L^2}^2 < \infty,$$

   and we record the simple ℓ^∞ bound

$$\sup_k |c_k(f)| \leq \frac{1}{\sqrt{2\pi}}\|f\|_{L^2}.$$

   We consider the candidate solution

$$u(t, x) := \sum_{k\in\mathbb{Z}} c_k(f)\, e^{-k^2 t}\, e^{ikx}.$$

   We claim that the above series converges absolutely for each (t, x) with $x \in \mathbb{R}$ and $t > 0$, and defines a function of class $C^\infty((0, \infty) \times \mathbb{R})$ which solves the heat equation.

For $\delta > 0$ set $\Omega_\delta := (\delta, \infty) \times \mathbb{R}$, and for each $m, n \in \mathbb{N}$, we prove the uniform convergence in Ω_δ of the series of functions

$$\sum_{k\in\mathbb{Z}} \partial_t^m \, \partial_x^n (c_k(f)\, e^{-k^2 t}\, e^{ikx}).$$

This implies that that our function u is indeed smooth on the same space. We note that

$$\begin{aligned} \sup_{(x,t)\in\Omega_\delta} \left|\partial_t^m \, \partial_x^n (c_k(f)\, e^{-k^2 t}\, e^{ikx})\right| &= \sup_{(x,t)\in\Omega_\delta} \left|(-k^2)^m \, (ik)^n \, c_k(f)\, e^{-k^2 t}\, e^{ikx}\right| \\ &\leq |k|^{2m+n}\, |c_k(f)|\, e^{-k^2\delta}. \end{aligned}$$

Thus we can prove the total convergence of the above series, since

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \sup_{(x,t)\in\Omega_\delta\times\mathbb{R}} \left|\partial_t^m \, \partial_x^n (c_k(f)\, e^{-k^2 t}\, e^{ikx})\right| &\leq \sum_{k\in\mathbb{Z}} |k|^{2m+n}\, |c_k(f)|\, e^{-k^2\delta} \\ &\leq \|\{c_k(f)\}\|_{\ell^\infty} \sum_{k\in\mathbb{Z}} |k|^{2m+n}\, e^{-k^2\delta} \leq \|f\|_{L^2} C(\delta, m, n) < \infty. \end{aligned}$$

Which demonstrates uniform convergence in Ω_δ. Hence, the potential solution u is smooth in Ω_δ for each $\delta > 0$. From the uniform convergence of the derivatives of all order, we also deduce the right to swap differential and infinite sum, which guarantees that u is a genuine solution.

2. Again, we apply Parseval's identity and find

$$\lim_{t\to 0} ||u(t,\cdot) - f||_{L^2}^2 = \lim_{t\to 0} \sum_{k\in\mathbb{Z}} \left|c_k(f)\, e^{-k^2 t} - c_k(f)\right|^2 = \lim_{t\to 0} \sum_{k\in\mathbb{Z}} |c_k(f)|^2 \underbrace{|e^{-k^2 t} - 1|^2}_{\to 0 \text{ for fixed } k.} = 0.$$

The last passage must be justified, for example using the Dominated Convergence Theorem, in fact the series on the RHS is the integral in $L^1(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \#)$ of $\phi_t(k) := |c_k(f)|^2 |e^{-k^2 t} - 1|^2$, which is dominated uniformly in t since

$$|\phi_t(k)| \leq 4|c_k(f)|^2 \in L^1(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \#),$$

and converges pointwise to 0 as $t \to 0$.

3. Let v be as in the question. We proceed as we did in the lecture notes, i.e, we show that u and v must have identical Fourier coefficients.

   Let $v(t,x) = \sum_{k\in\mathbb{Z}} d_k(t)\, e^{ikx}$ be the Fourier series representation of v. The dominated convergence theorem ensures that the Fourier coefficients

$$d_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(t,x)\, e^{-ikx}\, dx$$

are C^1 on $(0, \infty)$. Indeed, let $t_0 \in (0, \infty)$. We take as domination the constant function $||v||_{L^\infty([t_0-\delta, t_0+\delta]\times\mathbb{R})}$, for a small fixed $\delta > 0$ (we only need to ensure $t_0 - \delta > 0$)

and compute

$$\lim_{t\to t_0} d_k(t) = \lim_{t\to t_0} \frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)\, e^{-ikx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{t\to t_0} v(t,x)\, e^{-ikx}\, dx = d_k(t_0)$$

proving the continuity of d_k for arbitrary k.

Using again a constant as dominant for $\|\partial_t v\|_{L^\infty([t_0-\delta,t_0+\delta]\times\mathbb{R})}$, we can apply theorem A.32 (differentiation under the integral) and obtain

$$\frac{d}{dt} d_k(t)\Big|_{t=t_0} = \frac{d}{dt}\frac{1}{2\pi}\int_{-\pi}^{\pi} v(t,x)\, e^{-ikx}\, dx\Big|_{t=t_0} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \partial_t v(t,x)\, e^{-ikx}\, dx\Big|_{t=t_0} = c_k(\partial_t v(t_0,\cdot)),$$

ensuring the differentiability of $d_k(t)$ on $(0,\infty)$. An analogous argument to the one we used above shows that the d_k are in fact continuously differentiable on the same interval.

Using the continuous differentiability of the d_k, equation (1), and the behaviour of Fourier coefficients under differentiation, we derive

$$\frac{d}{dt} e^{k^2t} d_k(t) = e^{k^2t}\left(k^2 d_k(t) + c_k(\partial_t v(t,\cdot))\right) = e^{k^2t}\left(k^2 d_k(t) + c_k(\partial^2_{xx} v(t,\cdot))\right) = e^{k^2t}\left(k^2 d_k(t) + (ik)^2 \underbrace{c_k(v(t,\cdot))}_{=d_k(t)}\right) = 0.$$

Thus, $d_k(t) = \lambda_k e^{-k^2t}$, on $(0,\infty)$ for some $\lambda_k \in \mathbb{C}$, for all $k \in \mathbb{Z}$.

However, we know that both u, v satisfy equation (2), which implies

$$\lim_{t\to 0} ||v(t,\cdot) - u(t,\cdot)||_{L^2} = \lim_{t\to 0} ||v(t,\cdot) - f + f - u(t,\cdot)||_{L^2} \le \lim_{t\to 0} ||v(t,\cdot) - f||_{L^2} + ||u(t,\cdot) - f||_{L^2} = 0.$$

By Parseval's theorem

$$0 = \lim_{t\to 0} \sum_{k\in\mathbb{Z}} |(\lambda_k - c_k(f)) e^{-k^2t}|^2.$$

In particular this implies that for each fixed k we have

$$0 = \lim_{t\to 0} |\lambda_k - c_k(f)| e^{-2k^2t} = |\lambda_k - c_k(f)|^2 \Longleftrightarrow \lambda_k = c_k(f)$$

so we must have $\lambda_k = c_k(f)$, for all $k \in \mathbb{Z}$. This means that u and v have identical Fourier coefficients and are thus equal almost everywhere. But u, v are both continuous on $(0,\infty)\times\mathbb{R}$, so equal almost everywhere implies equality everywhere, completing the proof.

Solution of 9.4:

1. If we assume that the solution has the proposed form, then we infer

$$0 = (\partial_t^2 - \partial_x^2 - \lambda) \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx} = \sum_{k\in\mathbb{Z}} (u_k''(t) + (k^2 + |\lambda|)u_k(t))e^{ikx}.$$

   As the vectors $(e^{ikx})_{k\in\mathbb{Z}}$ are linearly independent, we find that for any index $k \in \mathbb{Z}$ the following homogeneous second order linear ODE must hold

$$u_k''(t) + (k^2 + |\lambda|)u_k(t) = 0 \quad \Rightarrow \quad u_k(t) = a_k \cos(\sqrt{k^2 + |\lambda|} \cdot t) + b_k \sin(\sqrt{k^2 + |\lambda|} \cdot t)$$

   for any $a_k, b_k \in \mathbb{C}$. Matching these coefficients with the boundary condition, we find that our solution must take the form

$$u(t, x) = \sum_{k\in\mathbb{Z}} u_k(t)e^{ikx},$$
$$u_k(t) := c_k(f) \cos(\sqrt{k^2 + |\lambda|} \cdot t) + \frac{c_k(g)}{\sqrt{k^2 + |\lambda|}} \sin(\sqrt{k^2 + |\lambda|} \cdot t).$$

2. We argue the exact same way as in the heat equation. Thus we want to show that each derivative of every summand is absolutely convergent in the supremum norm in $\mathbb{R} \times \mathbb{R}$, i.e. for any non-negative integers $\alpha, \beta \geq 0$ the following series is bounded

$$\sum_{k\in\mathbb{Z}} \left\| \partial_t^\alpha \partial_x^\beta u_k(t)e^{ikx} \right\|_\infty < +\infty. \tag{3}$$

   Indeed, as $f, g \in C^\infty_{\text{per}}$ we know from theorem 2.22 that for all non-negative integer $N \geq 0$ we have

$$\sum_{k\in\mathbb{Z}} |k|^N (|c_k(f)| + |c_k(g)|) < +\infty.$$

   One can quickly see that this bound also generalizes to

$$\sum_{k\in\mathbb{Z}} \left(k^2 + |\lambda|\right)^{\frac{N}{2}} (|c_k(f)| + |c_k(g)|) < +\infty.$$

   We are now in a position to prove (3) by

$$\sum_{k\in\mathbb{Z}} \left\| \partial_t^\alpha \partial_x^\beta u_k(t)e^{ikx} \right\|_\infty \leq \sum_{k\in\mathbb{Z}^*} |c_k(f)| \cdot \left(k^2 + |\lambda|\right)^{\frac{\alpha+\beta}{2}} + |c_k(g)| \cdot \left(k^2 + |\lambda|\right)^{\frac{\alpha+\beta}{2}-1} < +\infty. \tag{4}$$

3. Under the weaker assumption, we see that (4) still holds for $\alpha + \beta \leq 2$, thus ensuring C^2 regularity.

4. Using the chain rule, it is not difficult to verify that every function of the form $\phi(x-t)+\psi(x+t)$ constitutes a solution to the wave equation.

   For the converse direction we quickly see for any phases $\omega, \theta \in \mathbb{R}$ we have

$$\cos(\omega t)e^{i\theta x} = \frac{e^{i\omega t}+e^{-i\omega t}}{2}e^{i\theta x} = \frac{1}{2}\exp(i\theta x + i\omega t) + \frac{1}{2}\exp(i\theta x - i\omega t),$$
$$\sin(\omega t)e^{i\theta x} = \frac{e^{i\omega t}-e^{-i\omega t}}{2}e^{i\theta x} = \frac{1}{2}\exp(i\theta x + i\omega t) - \frac{1}{2}\exp(i\theta x - i\omega t).$$

   We can hence verify the claim by

$$\begin{aligned} u(t,x) &= \sum_{k\in\mathbb{Z}} \left(c_k(f)\cos(kt) + \frac{c_k(g)}{k}\sin(kt) \right) e^{ikx} \\ &= \sum_{k\in\mathbb{Z}} \left(c_k(f) + \frac{c_k(g)}{k} \right) \exp(ik(x+t)) + \sum_{k\in\mathbb{Z}} \left(c_k(f) - \frac{c_k(g)}{k} \right) \exp(ik(x-t)) \\ &= \phi_0(x+t) - \psi_0(x-t), \end{aligned}$$

   where we define ϕ_0 (and analogously (ψ_0)) by

$$\phi_0(t) := \sum_{k\in\mathbb{Z}} \left(c_k(f) + \frac{c_k(g)}{k} \right) \exp(ik(x+t)),$$

   the series is convergent in $C^2_{per}(\mathbb{R})$ under the decay assumptions on $c_k(f), c_k(g)$.

5. If the boundary value f is not continuously differentiable, then the wave equation has no smoothing properties for the same reason as the Schrödinger equation in Problem Set 8 has no smoothing properties, that is to say the kth time coefficient $\cos(\sqrt{k^2+|\lambda|}\cdot t)$ has no polynomial decay as $|k| \to +\infty$.

   On the other hand, if f has a continuous derivative, then u gains one derivative, regardless of the regularity of the boundary condition g. Indeed, this is a consequence of the time coefficient $\sin(\sqrt{k^2+|\lambda|}\cdot t)/\sqrt{k^2+|\lambda|}$, which has a polynomial decay of order 1.