EXAM SYLLABUS

This is the material you are expected to learn for both the oral and the written examination. As you can see it is a subset of what has been done in class and many proofs have been left out. Please be aware that also in the written examination you might be asked to provide a statement/proof/definition.

1. Hilbert Spaces

- Inner product spaces
  - Vector space: Definition.
  - Inner product space: Definition and properties.
  - Subspace is an inner product space: Statement.
  - Examples of inner product spaces: $L^2(X, \mu, \mathbb{R})$, $L^2(X, \mu, \mathbb{C})$, $\ell^2_{\mathbb{C}}$, and $\mathcal{C}([0, 1])$.
  - Norm: Definition.
  - Not every norm is induced by an inner product: Counterexample.
  - Cauchy-Schwarz inequality: Statement and **proof**.
  - Parallelogram law: Statement.
  - Polarization identities: Statements.
  - Inner product through the polarization identities.
  - Ptolemy's inequality: Statement.
  - Inner product continuity: Statement and **proof**. Continuity is not directly deducible from the one of both factors. Continuity is also a consequence of the polarization formulas.
  - Linear maps are isometries if and only if they preserve the inner product structure: Statement and **proof**.
- Normed vector spaces
  - Inner product spaces are normed vector spaces: Statement.
  - Examples of finite-dimensional vector spaces: $\mathbb{K}^d$ with p-norms ($1 \leq p < \infty$) and the maximum norm ($p = \infty$).
  - Examples of infinite-dimensional vector spaces: $\mathcal{C}([0, 1])$ with p-norms ($1 \leq p < \infty$) and the uniform norm ($p = \infty$).
  - L^p-spaces as measure spaces are complete normed vector spaces: Statement and **proof**.
  - Open ball: Definition.
  - Convexity: Definition.
  - Open balls are convex in a normed vector space: Statement and **proof**.
  - Interior points, open/closed sets, topology, topological vector space: Definitions.
  - Continuity of $+$ and $\cdot$ in a topological vector space: Result from the triangle inequality and homogeneity. Statement.
  - Convergence of a sequence: Definitions in the topological sense and in the metric space sense.
  - Completeness of a metric space, Cauchy sequence: Definitions.
  - Convergent sequences are Cauchy: Statement and **proof**.
  - Limit points: Definition.
  - Closed sets: A set is closed if and only if it contains all of its limit points. Statement.

  - Dense subsets: Definitions.
  - Equivalent norms: Definition.
  - In finite dimension, all norms are equivalent: Statement and **proof**.
  - In finite dimension, different norms induce the same topology: Statement.
  - Relation between the maximum norm and the 1- and 2-norms in $\mathbb{K}^d$: Statement and **proof**.
  - In infinite dimension, different norms do not necessarily induce the same topology: Counterexample.
- Hilbert spaces
  - Hilbert spaces can be viewed as a generalization of Euclideans spaces to infinite-dimensional settings: Statement.
  - Canonical norm and distance through an inner product: Definitions.
  - Hilbert space: Definitions.
  - Characterization by the validity of the parallelogram law: Statement.
  - Examples: $\mathbb{C}^d$, $L^2(X, \mu, \mathbb{C})$, $\ell^2_{\mathbb{C}}$ with their canonical scalar products, finite-dimensional inner product spaces.
  - Subspace is a Hilbert space if and only if it is closed. Proper dense subspaces are not Hilbert spaces: Statements and **proofs**.
  - Inner product spaces are not necessarily complete: Counterexample.
  - Orthogonality: Definition. Relation between orthogonality and the norm in an inner product space.
  - Projection operator, Gram-Schmidt orthogonalization process and orthonormal basis: Definitions.
- Basis of a Hilbert space
  - Algebraic basis and finite-dimensional vector spaces: Definitions.
  - Separability: Definition.
  - Examples of separable topological spaces: $\mathbb{R}^d$, $\mathbb{C}^d$, compact metric spaces, $\mathcal{C}(K)$, L^p-spaces.
  - Orthonormal system: Definition. Bessel inequality and Parseval's identity: Statements.
  - Hilbert basis: Definition.
  - Equivalence of Hilbert and algebraic bases in finite dimensions, distinction in infinite dimensions.
  - Completeness criterion: Statement and **proof**.
  - Existence of a basis: Statement.
  - Separable complex Hilbert spaces are isometric to $\ell^2_{\mathbb{C}}$: Statement and **proof**.
- Closest point property, projections
  - Projections on closed vector subspaces and closed convex sets: Statements and **proofs**.
  - Orthogonal space is not trivial: Statement and **proof**. Importance of the closedness hypothesis.
  - Projection over finite-dimensional and separable closed subspaces: Statements and **proofs**.
  - Orthogonal complement: Definition. Closedness, non-triviality of the orthogonal complement and trivial intersection with the linear space: Statements and **proofs**.
  - Orthogonal decomposition: Statement and **proof**.
- Linear operators and continuous functionals
  - Linear, bounded and unbounded operators, functionals, $L(X, Y)$ and continuous dual space: Definitions.

  - Example of unbounded operator: the derivative operator.
  - Operator norm: Definition. Equivalent formulations of the norm: Statement.
  - Equivalence between boundedness and continuity: Statement.
  - Riesz Representation Theorem: Statement and **proof**.
  - Isomorphism between a Hilbert space and its dual: Statement.

2. Fourier Series

- Definitions and main properties
  - Fourier coefficient: Definition. Well-posedness in L^1 and L^2: Statement and **proof**. Fourier coefficients are bounded linear functionals on L^2: Statement.
  - Fourier partial sums: Definition.
  - Fourier Basis Theorem: Statement.
  - Convergence of the Fourier partial sums in L^2, Parseval's identity and its inner product version in L^2: Statement.
  - Expressions of Fourier coefficients and partial sums using sine and cosine, and simplifications based on function parities: Statements and **proofs**.
  - Examples: Fourier coefficients of trigonometric functions and trigonometric polynomials.
  - Examples: Computation of series through Parseval's identity.
  - Equivalence between real-valuedness of the function and conjugation symmetry of the Fourier coefficients: Statement.
- Series in Banach spaces
  - Convergence criteria of series in Hilbert spaces and Pythagoras' Theorem: Statements.
  - Completeness and convergence in $\mathcal{C}_b^m(\Omega;\mathbb{C})$: Statement and **proof**.
- Regularity and asymptotic behavior of Fourier coefficients
  - Fourier coefficients of the derivative: Statement and **proof**. Generalization to $\mathcal{C}^h$ functions: Statement.
  - Asymptotic behavior of Fourier coefficients in $\mathcal{C}^1$: Statement and **proof**. Generalization to $\mathcal{C}^h$ functions: Statement.
  - Uniform convergence of the Fourier partial sum of $\mathcal{C}^1$ functions: Statement and **proof**. Generalization to $\mathcal{C}^h$ functions: Statement.
  - Summability of Fourier coefficients implies regularity of the function and uniform convergence of the Fourier partial sums along the derivatives: Statement.
- Pointwise convergence of Fourier series
  - Locality of the pointwise convergence of the Fourier partial sum: Statement and **proof**.
  - Integral of the Dirichlet Kernel: Statement and **proof**. Explicit formula: Statement.
  - Riemann-Lebesgue Lemma: Statement and **proof**.
- Overview of convergence (tables)
  - Relation between the modes of convergence
  - Nested classes of functions
  - Size of the Fourier coefficients and convergence of the Fourier partial sums
- Heat equation
  - Uniqueness of solution: Statement.
  - Non-existence in the past: Statement and **proof**.

3. Fourier Transform

- Fourier transform in $L^1(\mathbb{R}^d)$
  - Fourier transform: Definition. Well-posedness in L^1 and properties: Statements and **proofs**.
  - Translation, modulation, dilatation, convolution formulas: Statements and **proofs**.
  - Examples: Exponential envelope function $e^{-|x|}$ and characteristic function $\mathbf{1}_{[-1,1]}$: Statements and **proofs**. Normal Gaussian distribution Φ_d: Statement.
  - Fourier transform of a radial function is radial: Statement and **proof**.
  - Parity and valuedness of the Fourier transform: Statements and **proofs**.
- Space of Schwartz functions
  - Fourier transform of partial derivatives: Statement and **proof**.
  - Derivative of the Fourier transform: Statement and **proof**.
  - Fourier transform of the normal Gaussian distribution through an initial value problem: Statement and **proof**.
  - Example: Fourier transform of $xe^{-|x|}$: Statement and **proof**.
  - Schwartz space: Definition. Inclusion in L^p-spaces and growth rate of Schwartz functions: Statements.
  - Closedness under partial derivation and multiplication by polynomials: Statement.
  - Strict inclusion of smooth, compactly supported functions in the Schwartz space: Statement and counterexample for equality.
  - Semi-norm and norm on Schwartz space: Statement and **proof**.
  - Schwartz functions have smooth Fourier transforms, Fourier transform of the derivatives and derivatives of the Fourier transform: Statements.
  - Schwartz functions have Schwartz Fourier transforms: Statement.
  - Inversion formula in $\mathcal{S}(\mathbb{R}^d)$ and relaxation to L^1: Statements.
  - Shift formula: Statement.
  - Injectivity of the Fourier transform: Statement.
  - Examples: $u * u = u$ implies $u = 0$ almost everywhere, Fourier transforms of $\frac{1}{1+x^2}, \frac{1}{(1+x^2)^2}$ and $\frac{\sin(2x)}{1+x^2}$, and equation $u * u(x) = \frac{2}{1+x^2}$.
- Fourier transform in $L^2(\mathbb{R}^d)$
  - The Fourier transform is an isometry on L^2: Statement.
  - Fourier transform in L^2: Definition. Well-posedness: Statement.
  - Plancherel's identity: Statement and **proof**.
  - Fourier transform of partial derivatives: Statement.
- Overview of results & properties (tables)
  - Fourier transform of Schwartz class functions
  - Correspondence between operations in $\mathcal{S}(\mathbb{R}^d)$
  - Important Fourier transforms

4. Spectral theory

- Compact operators
  - Definition of compact operator. Characterization of compact operators.
  - Definition of finite rank operators and compactness.
  - The space of compact operators is a closed subspace of the space of linear operators
  - Example: integral operators.

  - Definition of adjoint operator. Theorem of existence and uniqueness of the adjoint operator.
  - Proposition: an operator is compact if and only if its adjoint is compact
- Fredholm theory and spectral theorem
  - Definition: eigenvalue and eigenvectors
  - Theorems: Fredholm alternative I, II, III and IV (Statements and **proof** of Fredholm Alternative III).
  - Definition of resolvent and spectrum
  - Open mapping theorem and Banach fixed point theorem
  - Theorem: structure theorem of the spectrum of compact operators
  - Spectral theorem