1. Goal and Motivation

The goal of this lecture is use the Spectral Theorem for compact operators to prove the following

Theorem 1.1. *Let $V \in C^\infty(\mathbb{R})$ be a nonnegative smooth function and $I = (0,1)$. There exist*

- *an sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\varepsilon_k \to +\infty$,*
- *a sequence $\psi_k \in C^\infty(I) \cap C(\overline{I})$ such that*

$$(1)\qquad -\psi_k''(x) + V(x)\psi_k(x) = \varepsilon_k \psi_k(x) \text{ for all } x \in I, \quad \psi_k(0) = \psi_k(1) = 0,$$

and $\{\psi_k\}_{k\in\mathbb{N}}$ is an Hilbert basis of $L^2(I)$.

Remark 1.2. In the case $V \equiv 0$ we can solve explicitly (1) and find

$$\varepsilon_k = \pi^2(k+1)^2 \quad \text{and} \quad \psi_k(x) = \tfrac{1}{\sqrt{\pi}}\sin(\pi(k+1)x).$$

By Fourier theory, $\{\frac{1}{\sqrt{\pi}}\sin(\pi(k+1)x)\}_{k\in\mathbb{N}}$ is an Hilbert basis of $L^2(I)$.

Theorem 1.1 is a far reaching generalisation of this example as, for a general V, there is no hope to give formulas for the ψ_k's.

Before starting the proof let us show a typical application of Theorem 1.1.

Example 1.3. Consider the Schrödinger equation

$$i\partial_t w = Hw, \quad H := -\frac{d^2}{dx^2} + V(x), \quad w(t,0) = w(t,1) = 0, \quad w(0,x) = f(x),$$

where $w(t,x)\colon \mathbb{R}\times\overline{I} \to \mathbb{C}$ and $f \in L^2(I)$[1]. By Theorem 1.1 there is an orthonormal basis $\{\psi_k\}$ such that $H\psi_k = \varepsilon_k\psi_k$, for some "energies" ε_k. If we express then the initial state $f(x) = \sum_k \langle f, \psi_k\rangle \psi_k(x)$ we find that the solution w must be given by

$$w(t,x) = \sum_k \langle f, \psi_k\rangle e^{-i\varepsilon_k t}\psi_k(x),$$

where the series converges at least in the L^2 sense (and much better if f is assumed to be more regular).

Notice that if $f = \psi_N$, then $w(t,x) = e^{-i\varepsilon_N t}\psi_N(x)$, and in particular $|w|$ is constant, this is why the ψ_k's are called *steady states.*

2. Proof of Theorem 1.1

2.1. **The Spectral Theorem.**

As anticipated, the proof of Theorem 1.1 relies on

Theorem 2.1 (Spectral Theorem)**.** *Let $\mathcal{H}$ be an Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$ be compact and symmetric. Then there is a real infinitesimal sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ and an Hilbert basis $\{e_k\}_k \in \mathbb{N}$ of $\mathcal{H}$ such that*

$$Tv = \sum_k \lambda_k \langle v, e_k\rangle e_k \quad \text{for all } v \in \mathcal{H}.$$

Proof. See the Lecture Notes. ☐

[1] In Quantum Mechanics, $|w(t,x)|^2\,dx$ represents the probability of finding at time t in the interval $[x, x+dx]$ a quantum particle, prepared at time $t = 0$ in the state $f(x)$, subject to the Hamiltonian H and confined to live in a box $x \in I$.

2.2. **An auxiliary Hilbert space.** Define the scalar product

$$\langle u, v\rangle_V := \int_0^1 u'(x)v'(x) + V(x)u(x)v(x)\,dx, \qquad u, v \in C^1_c(I). \tag{2}$$

Notice that for all $0 \le x < y \le 1$ and $u \in C^1_c$ it holds

$$|u(x) - u(y)| = \Big|\int_x^y u'(s)\,ds\Big| \le \int_x^y |u'| \le \|u'\|_{L^2(I)}|x-y|^{\frac12} \le \|u\|_V|x-y|^{\frac12}.$$

So in particular

$$\|u\|_{L^\infty(I)} + \sup_{x,y\in I}\frac{|u(x)-u(y)|}{|x-y|^{\frac12}} \le 2\|u\|_V. \tag{3}$$

We also define the Hilbert space $(\mathcal{H}, \langle u, v\rangle_V)$ where

$$\mathcal{H} := \overline{C^1_c(I)}^{\|\cdot\|_V}.$$

Thanks to (3), we have $\mathcal{H} \subset C^{\frac12}(\overline{I})$ and, concretely, $u \in \mathcal{H}$ if and only if there is a sequence $\{u_j\} \subset C^1_c(I)$ such that

$$\{u_j\} \text{ is Cauchy with respect to } \|\cdot\|_V \text{ and } u_j \to u \text{ in } L^2(I).$$

The key observation is the following

Proposition 2.2. *The inclusion $\mathcal{H} \subset L^2(I)$ is compact: if $\{w_j\} \subset \mathcal{H}$ is bounded, then a subsequence converges strongly in $C(\overline{I})$ and thus in $L^2(I)$.*

Proof. For each $j \in \mathbb{N}$, by definition, we can write $w_j = \tilde{w}_j + \xi_j$ with

$$\tilde{w}_j \in C^1_c(I) \text{ and } \|\xi_j\|_V \le 2^{-j}.$$

If $M := \sup_j \|w_j\|_V$ then $\|\tilde{w}_j\|_V \le M + 1$ and so by (3) we have

$$\|\tilde{w}_j\|_{L^\infty(I)} + \sup_{x,y\in I}\frac{|\tilde{w}_j(x)-\tilde{w}_j(y)|}{|x-y|^{\frac12}} \le 2M + 2.$$

Thus the family of continuous functions $\{\tilde{w}_j\} \subset C(\overline{I})$ is equibounded and equicontinuous, so — by Ascoli-Arzelá — has a subsequence $\tilde{w}_{j'} \to \bar{w}$ uniformly in $C(\overline{I})$. By (3), we also have $\xi_{j'} \to 0$ uniformly, so in conclusion $w_{j'} \to \bar{w}$ in $C(\overline{I})$. □

We conclude remarking that every $u \in \mathcal{H}$ must vanish at the endpoints if I.

Exercise 2.3. By approximation, show that $\sin(\pi k x) \in \mathcal{H}$ for all $k \in \mathbb{N}$. More generally, show that if $v \in C^1(I) \cap C(\overline{I})$ with $\int_I \dot{v}^2 < \infty$ and $v(0) = v(1) = 0$, then $v \in \mathcal{H}$.

2.3. **Weak formulation of** (1)**.** Given $f \in L^2(I)$, we say that $u \in \mathcal{H}$ solves (1) weakly if

$$\langle u, w\rangle_V = \langle f, w\rangle_{L^2} \qquad \text{for all } w \in \mathcal{H}. \tag{4}$$

Let us motivate this definition, assume for a second that u, f are smooth and

$$-u'' + Vu = f.$$

If we multiply both sides of the equation by $w \in C^1_c(I)$ and integrate over I we find

$$-\int_I u''w + Vuw = \int_I fw.$$

Now we can integrate by parts the first integral (the boundary term vanish!) and find

$$\langle u, w\rangle_V = \int_I u'w' + Vuw = \int_I fw = \langle f, w\rangle_{L^2}.$$

The following proposition will be crucial to "go back" from the weak formulation (4) to the classical one (1).

Proposition 2.4. *Let $f \in L^2(I)$ and $u \in \mathcal{H}$ satisfying* (4)*. Then*

$$k^2 c_k(u) = c_k(f - Vu) \text{ for all } k \in \mathbb{N}.$$

Proof. Fix $k \in \mathbb{N}$. By the Exercise above $w := \frac{1}{\sqrt{\pi}} \sin(k\pi x) \in \mathcal{H}$. Write $u = \tilde{u}_j + \xi_j$ with $\|\xi_j\|_V \leq 2^{-j}$. Then we have

$$\begin{aligned} c_k(f) &= \langle f, w\rangle_{L^2(I)} = \langle u, w\rangle_V = \langle \tilde{u}_j, w\rangle_V + \langle \xi_j, w\rangle_V \\ &= \int_I \tilde{u}_j' w' + \int_I V\tilde{u}_j w + \langle \xi_j, w\rangle_V \\ &= k^2 \int_I \tilde{u}_j w + \int_I V\tilde{u}_j w + \langle \xi_j, w\rangle_V. \end{aligned}$$

Passing to the limit as $j \to \infty$ we find

$$c_k(f) = k^2 c_k(u) + c_k(Vu).$$

□

2.4. **Existence if weak solutions and the operator T.** We start solving (4).

Lemma 2.5. *Given $f \in L^2(I)$ there exists a unique $u \in \mathcal{H}$ such that* (4) *holds.*

Proof. The linear functional

$$F\colon \mathcal{H} \to \mathbb{R}, \quad F(w) := \langle f, w\rangle_{L^2(I)},$$

is bounded in $\mathcal{H}$ since for all $w \in \mathcal{H}$ it holds

$$|F(w)| \leq \|f\|_{L^2(I)} \|w\|_{L^2(I)} \leq 2\|f\|_{L^2(I)} \|w\|_V.$$

Thus, by Riesz' representation Theorem in $\mathcal{H}$, there is a unique $u \in \mathcal{H}$ such that

$$\langle u, w\rangle_V = \langle f, w\rangle_{L^2(I)} \text{ for all } w \in \mathcal{H}.$$

This means exactly that there is a unique solution u to (4). □

This Lemma defines a bounded linear operator $\tilde{T}\colon L^2(I) \to \mathcal{H}$ as

$$\tilde{T}\colon L^2(I) \ni f \mapsto u \in \mathcal{H} \text{ which solves (4) for } f.$$

Then we define the compact operator $T\colon L^2(I) \to L^2(I)$ as the composition

$$T\colon L^2(I) \xrightarrow{\tilde{T}} \mathcal{H} \hookrightarrow L^2(I).$$

This operator is compact because of Proposition 2.2.

Let us check that T is symmetric:

$$\langle f, Tg\rangle_{L^2(I)} = \langle Tf, Tg\rangle_V = \langle Tg, Tf\rangle_V = \langle g, Tf\rangle_{L^2(I)}.$$

2.5. **Conclusion of the proof.** Since T is compact and symmetric on $L^2(I)$, the Spectral Theorem 2.1 applies: there is a sequence of eigenvalues $\lambda_j \to 0$ and a complete orthonormal system of eigenfunctions $\psi_j \in L^2(I)$ such that

$$\lambda_j \langle \psi_j, w\rangle_V = \langle \psi_j, w\rangle_{L^2(I)}, \text{ for all } w \in \mathcal{H}, j \in \mathbb{N}.$$

Notice that $\mathrm{ran}(T) \subset \mathcal{H}$, so we have $\psi_j \in \mathcal{H}$. Taking $w = \psi_j$ we also find that $\lambda_j > 0$ for all $j \in \mathbb{N}$.

In order to conclude the proof of Theorem 1.1 we must prove that each ψ_j is in fact C^∞ in I. This is done by the so called "bootstrap" procedure, applying repeatedly Proposition 2.4.

First we have $\psi_j - V\psi_j \in L^2(I)$, so

$$\{k^2 c_k(\psi_j)\}_k \in \ell^2(\mathbb{N}).$$

This immediately gives $\psi_j \in C^1(I)$ and $\psi_j' \in L^2(I)$, so $\psi_j' - (V\psi_j)' \in L^2(I)$. Then we find

$$\{k^2 c_k(\psi_j - V\psi_j)\}_k \in \ell^2(\mathbb{N}).$$

Again Proposition 2.4 gives

$$\{k^4 c_k(\psi_j)\}_k \in \ell^2(\mathbb{N}),$$

which in turn implies $\psi_j \in C^2(I)$ and $\psi_j'' \in L^2(I)$. Thus $\psi_j'' - (V\psi_j)'' \in L^2(I)$ and we find

$$\{k^4 c_k(\psi_j - V\psi_j)\}_k \in \ell^2(\mathbb{N}).$$

and so

$$\{k^6 c_k(\psi_j)\}_k \in \ell^2(\mathbb{N}).$$

It is clear that this procedure never stops and shows

$$k^M c_k(\psi_j) \in \ell^2(\mathbb{N}) \text{ for all } M \geq 1,$$

so $\psi_j \in C^\infty(I)$. Finally, by Proposition 2.4 we find

$$c_k(-\lambda_j \psi_j'' + V\psi_j - \psi_j) = 0 \text{ for all } k \in \mathbb{N},$$

and the only possibility is that each ψ_j solves

$$\lambda_j \psi_j''(x) + V(x)\psi_j(x) = \psi_j(x) \qquad \text{for all } x \in I.$$

Theorem 1.1 follows setting $\varepsilon_j := 1/\lambda_j$.

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