TOPOLOGY SPRING 2024
SERIE 10

(1) Let $B = \{1/n \mid n \geq 1\} \subset \mathbf{R}$. Define a topology $\mathscr{T}^*$ on $\mathbf{R}$ with basis $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$, where

$$\mathscr{B}_1 = \{]a, b[\mid a < b \text{ in } \mathbf{R}\}$$
$$\mathscr{B}_2 = \{]a, b[\setminus B \mid a < b \text{ in } \mathbf{R}\}$$

i.e., a set $U \subset \mathbf{R}$ is open for $\mathscr{T}^*$ if and only if it is an arbitrary union of sets in $\mathscr{B}$.

(a) Show that $\mathscr{T}^*$ is indeed a topology, and that $\mathbf{R}$ is Hausdorff with this topology.
(b) Let $A = \{0\}$. Show that the sets A and B are closed in $(\mathbf{R}, \mathscr{T}^*)$.

We now suppose that U and V are open sets with $0 \in U$, $B \subset V$, and that $U \cap V = \emptyset$.

(c) Show that there exist $a < b$ such that $a < 0 < b$ and $]a, b[\setminus B \subset U$.
(d) Show that there exists an integer $n \geq 1$ such that $1/n \in]a, b[$.
(e) Show that $1/n \in V$ and that there exist $c < 1/n < d$ such that $]c, d[\subset V$.
(f) Show that there exists $x \in \mathbf{R}$ such that

$$\frac{1}{n+1} < x < \frac{1}{n}, \qquad x > c.$$

(g) Show that $x \in U$ and that $x \in V$.
(h) Conclude that $(\mathbf{R}, \mathscr{T}^*)$ is not normal. (In fact, it is not even *regular*, where a space is called regular if it is Hausdorff and for any $x \in X$ and $B \subset X$ closed not containing x, there are disjoint open sets U and V with $x \in U$ and $B \subset V$.)

(2) Let X be a normal topological space. Let

$$\mathscr{F} = \{f : X \to [0, 1] \mid f \text{ is continuous}\}.$$

For $f \in \mathscr{F}$, let $X_f = [0, 1]$, and let

$$\varphi : X \to \prod_{f \in \mathscr{F}} X_f$$

be the map defined by

$$\varphi(x) = (f(x))_{f \in \mathscr{F}}.$$

We denote $Y = \varphi(X)$.

(a) Show that φ is injective.
(b) Show that φ is continuous when the product space has the product topology.
(c) Let $y = \varphi(x)$ be an element of Y. Show that a fundamental system of open neighborhoods of y in Y is given by the sets

$$\{\varphi(z) \mid z \in X \text{ satisfies } |f_j(z) - f_j(x)| < \varepsilon_j \text{ for all } j \in J\},$$

where $f_j \in \mathscr{F}$ for all $j \in J$, J runs over finite sets and ε_j runs over positive reals for all $j \in J$.

(d) Let U be open in X and let $x_0 \in U$. Show that there exists an open neighborhood V of x_0 such that $V \subset \overline{V} \subset U$, and a function $g \in \mathscr{F}$ such that

$$\{z \in X \mid g(z) > 1/2\} \subset U.$$

(e) Deduce that the map $\varphi\colon X \to Y$ is a homeomorphism. (Hint: show using the previous questions that the image by φ of an open set in X is open in Y.)

(f) Deduce that X is homeomorphic to a subspace of a compact space.

(3) Let X be a normal space. For a continuous function $f\colon X \to \mathbf{C}$, we define the *support* of f, denoted $\mathrm{Supp}(f)$, to be

$$\mathrm{Supp}(f) = \overline{f^{-1}(\mathbf{C} \setminus \{0\})}$$

(the closure of the set of x where $f(x) \neq 0$).

Given a finite family $(U_i)_{1\leq i\leq k}$ of open subsets of X whose union is X, a *partition of unity* subordinate to this covering is a finite family $(f_i)_{1\leq i\leq k}$ of continuous functions $f_i\colon X \to [0,1]$ such that

- We have $\mathrm{Supp}(f_i) \subset U_i$ for all i.
- We have

$$\sum_{i=1}^{k} f_i(x) = 1$$

for all $x \in X$.

The goal of the exercise is to show that there always exists such a partition of unity.

(a) Show that given a finite open covering $(U_i)_{1\leq i\leq k}$, for $1 \leq i \leq k$ we can find $V_i \subset U_i$, open, with $\overline{V}_i \subset U_i$, such that $(V_i)_{1\leq i\leq k}$ is a covering of X. (Hint: show by induction on $j \leq k$ that there are V_i, $i \leq j$, with $\overline{V}_i \subset U_i$, such that $(V_1, \ldots, V_j, U_{j+1}, \ldots, U_k)$ cover X.)

(b) Show that there are coverings $(W_i)_{1\leq i\leq k}$ and $(V_i)_{1\leq i\leq k}$ and functions $g_i\colon X \to [0,1]$ such that

$$\overline{W}_i \subset V_i \subset \overline{V}_i \subset U_i,$$

and

$$g_i(x) = \begin{cases} 1 & \text{if } x \in W_i, \\ 0 & \text{if } x \in X \setminus V_i. \end{cases}$$

(c) Show that $\mathrm{Supp}(g_i) \subset U_i$ and that

$$\sum_{i=1}^{k} g_i(x) > 0$$

for all $x \in X$.

(d) Deduce the existence of a partition of unity subordinate to (U_i).

(4) Let X be a compact Hausdorff topological manifold of dimension $d \geq 1$. The goal of this exercise is to show that there exists some integer $m \geq 1$ and a compact subset $C \subset \mathbf{R}^m$ homeomorphic to X.

(a) Show that there exist a finite covering $(U_i)_{1\leq i\leq k}$ of X by open sets such that for every i, there is an homeomorphisms $\varphi_i\colon U_i \to W_i$, where $W_i \subset \mathbf{R}^d$ is open.

(b) Explain why there exists a partition of unity $(f_i)_{1\leq i\leq k}$ subordinate to (U_i) (as defined in the previous exercise). Show that the functions $g_i\colon X \to \mathbf{R}^d$ defined by

$$g_i(x) = \begin{cases} f_i(x)\varphi_i(x) & \text{if } x \in U_i, \\ 0 & \text{if } x \in X \setminus \operatorname{Supp}(f_i) \end{cases}$$

are continuous (where the support of f_i is defined also in the previous exercise).

(c) Show that the map $\varphi\colon X \to \mathbf{R}^k \times \mathbf{R}^{dk}$ defined by

$$\varphi(x) = (f_1(x), \ldots, f_k(x), g_1(x), \ldots, g_k(x))$$

is injective. (Hint: if $\varphi(x) = \varphi(y)$, show that there exists i such that $x \in U_i$ and $y \in U_i$.)

(d) Show that φ is continuous and that it defines a homeomorphism

$$\varphi\colon X \to \varphi(X) \subset \mathbf{R}^{k+dk}.$$